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On Logical Connectives and Quantifiers as Adjoint Functors

Master Thesis in
Mathematics

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Abstract

This thesis deals with the issue of treating logical connectives, quantifiers and equality in categorical terms, by means of adjoint functors combined into the notion of hyperdoctrine, introduced by Francis William Lawvere in 1969.

After proving the general Theorem of Soundness and Completeness for the intuitionistic predicate logic with equality with respect to hyperdoctrines, we formulate instances of such categorical models by using H-valued sets and Kleene realizability, in order to produce easily models and countermodels for logical formulas.

To my overseas

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Introduction

The main purpose of the thesis is the proof of the Theorem of Soundness and Completeness for predicate logic with equality with respect to the categorical semantics of Lawvere's hyperdoctrines. Categorical logic was introduced during the 1960s and 1970s by Lawvere and can roughly be defined as the treatment of logic in a category-theoretic framework. In this framework the semantics of logical connectives and quantifiers is defined by means of adjoint functors, thus it differs from the well-known and extensively used semantics for intuitionistic and classical logic, namely complete Heyting and Boolean algebras respectively. More precisely, category theory, and thus categorical logic, allows an extremely general treatment of algebraic structures, hence we should more properly assert that categorical semantics extends the canonical algebraic semantics. We should also underline that categorical logic allows a full constructive direct proof of the Theorem of Soundness and Completeness, whereas this is no longer possible with a canonical algebraic approach to logic.

Next to a close attention to the mathematical aspects of the issue, also a sketch of the underlying historical events can be found in the thesis. There are two main reasons why we opt for such a choice. First, we do think that mathematical constructions go hand-in-hand with historical facts, and thus that the former can more accurately explain and be explained by the latter. Second, we have to recognize that categorical logic is a really wide and organic subject, despite its recent formulation, and that the part we are going to take into consideration is just a drop in the bucket. However, it represents the starting point of many other developments in research, which can be depicted, if not in their many formal aspects, at least by giving a clue of their historical importance. Exactly for the same reasons, the historical and the formal treatments of the issue are not kept separated, but rather they weave together drawing strength one from the other. On facing such a project, one or two things have to be born in mind, though. In the first place, we should underline the lack of an accepted historiography on the subject, which is motivated by the recent historical developments of category theory and categorical logic. Secondly, the authors of the very few publications dealing with this matter are most of the times the leading actors of the events they are reporting, therefore an external point of view cannot be provided yet.

The thesis can be in some sense divided into four parts. At the beginning, an introduction of the basic concepts of category theory is imperative: as we already said, categorical logic can in fact be thought of as a branch of category theory. Afterwards, the basics of categorical logic are introduced: logical connectives and quantifiers, as well as the equality in a language, are shown to be characterized by an essentially adjoint-functorial nature. Finally, the proof of the Theorem of Soundness and Completeness for categorical predicate logic with identity is presented. In the very end, one can find a sketchy introduction to the basics of Kleene's realizability theory, presented in the form of hyperdoctrine.

Chapter 1, Chapter 2 and Chapter 3 introduce the basic notions of category theory. First of all, the concept of category is introduced. Different categories can be related one to another via functors. Sometimes different functors behave in a really similar fashion, so it is possible to pass from one to another via natural transformations. More importantly, the way certain adjoint functors can be composed often spots pivotal patterns in the mathematical landscape: undoubtedly, adjunction turns out to be the most important construction of category theory and as such it weaves together the many different concepts and arguments this thesis tackles. Whereas adjunction describes relationships between categories, limits (and their dual, colimits) provide an internal view of categories and formalize many mathematical constructions such as products, sums, pullbacks and many others under a unique notion.

Chapter 4 and Chapter 5 deal with preorders and Heyting, Boolean and Lindenbaum algebras. They are so intimately related one to each other that at some points notions described in the latter can be necessary to understand those of the former. Indeed, algebras and preorders have at their cores the same mathematical structures, which can be described by means of category theory. We decided to present the matters following this order because some observations at the end of Chapter 5 are hints that prepare the reader for the passage from a an algebraic treatment of logic to a category-theoretic one.

Chapter 6 and Chapter 7 are in fact the first real chapters about categorical logic as such. The main concern of these two chapters is to "translate" in a purely categorical framework logical connectives and quantifiers: whereas the remarks of Chapter 5 give the idea for a categorical introduction of all connectives, this is not the case for quantifiers. The categorical treatment of logical quantifiers passes through the formalization of the notion of hyperdoctrine. Hyperdoctrines provide in addition the means to formalize categorically the identity in logical languages.

Chapter 8 represents the core of the thesis: the proof of the Theorem of Soundness and Completeness for predicate logic with equality is provided. Besides, since categorical toolkit allows a very general treatment of mathematical concepts, the proof of the Theorem is also presented in a particular case, which will be useful for the construction of a countermodel for the well-known non-intuitionistically valid formula $\forall x_1 \forall x_2.(x_1 = x_2 \vee x_1 \neq x_2)$.

Chapter 9 is all devoted to an example of application of the categorical notions so far introduced. More specifically, we will propose a treatment of Kleene's theory of realizability for predicate logic in category-theoretic terms and we will use this to show that the rules of classical logic, unlike those of intuitionistic logic, are not valid in Kleene's realizability.

Chapter 1

Basic concepts of category theory

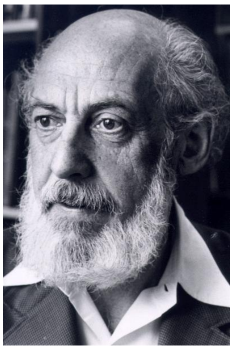
The main purpose of this chapter is introducing the basic notions of category theory, such as category, functor and natural transformation. A category is a formalization of the most general idea of a mathematical structure: inside a given structure, we can find the mathematical objects that belong to it, as well as some mathematical connections between them. Different structures can be somehow connected one to each other: this idea gives rise to the notion of functor. Besides, the ways of connecting different structures can turn out to have something in common between them: this is formalized in the notion of natural transformation.

1.1 Halfway between category theory and logic

Categorical logic is logic in the setting of category theory. Exact though this definition might be, it is undoubtedly much too poor and shallow to be accepted with no further comments. We find more instructive to look at categorical logic as a "generalization of the algebraic encoding of propositional logic to first-order, higher-order and other logics" ([MR11], pg 1.): in the same way (either intuitionistic or classical) propositional logic can be encoded by algebraic logic by means of Lindenbaum-Tarki algebras, so categorical logic encodes first-order and higher order logics by means of categories with specific properties and structures.

It becomes apparent now the reasons why the history and development of categorical logic is deeply related to the birth and evolution of category theory. Unavoidably, we have to begin by introducing the basic notions of category theory, in order to pass on later to treat categorical logic.

Category theory appeared officially in 1945 in the paper *General Theory of Natural Equivalences* ([ME45]) by Samuel Eilenberg (September 30, 1913 – January 30, 1998) and Saunders MacLane (August 4, 1909 – April 14, 2005). In MacLane's words, the paper was "off-beat" and "far out" ([Mac02], pg 130).



S. Eilenberg

S. MacLane

The initial idea of the two mathematicians was to provide an autonomous framework for the concept of **natural transformation**, which they came across during their studies and whose generality, pervasiveness and usefulness became soon clear to both of them. For this reason, their initial project turned into that one of devising an axiomatic system in which natural transformations would arise naturally from a stable and self-consistent theoretic grounding. After they realized that a natural transformation is nothing but a family of maps providing a sort of "deformation" between two "collections of interrelated entities" within a given structure, they introduced what are now called **functors**, which played the role of the "deformation" of a natural transformation. The "collection of interrelated entities" was soon formalized through the definition of **category**. Noteworthy is the fact that MacLane and Eilenberg explicitly avoided using a set-theoretical terminology and notation.

Out of sheer historical interest, we report here the very first definition of category ever, as it appeared in MacLane and Eilenberg's paper.

Definition 1.1. *A **category** $\mathcal{A} = (A, \alpha)$ is an aggregate of abstract elements A (for example, groups), called the **objects** of the category, and abstract elements α (for example, homomorphisms), called **mappings** of the category. Certain pairs of mappings α_1, α_2 determine uniquely a product mapping $\alpha = \alpha_2\alpha_1 \in \mathcal{A}$ subject to the axioms C1, C2, C3 below. Corresponding to each object $A \in \mathcal{A}$ there is a unique mapping, denoted by e_A or by $e(A)$, and subject to the axioms C4 and C5. The axioms are:*

C1 *The triple product $\alpha_3(\alpha_2\alpha_1)$ is defined if and only if $(\alpha_3\alpha_2)\alpha_1$ is defined. When either is defined, the associative law*

$$\alpha_3(\alpha_2\alpha_1) = (\alpha_3\alpha_2)\alpha_1$$

holds. This triple product will be written as $\alpha_3\alpha_2\alpha_1$.

C2 *The triple product $\alpha_3\alpha_2\alpha_1$ is defined whenever both products $\alpha_3\alpha_2$ and $\alpha_2\alpha_1$ are defined.*

*A mapping $e \in \mathcal{A}$ will be called an **identity** of $\mathcal{A}$ if and only if the existence of any product $e\alpha$ or βe implies that $e\alpha = \alpha$ and $\beta e = \beta$.*

C3 *For each mapping $\alpha \in \mathcal{A}$ there is at least one identity $e_1 \in \mathcal{A}$ such that αe_1 is defined, and at least one identity $e_2 \in \mathcal{A}$ such that $e_2\alpha$ is defined.*

C4 *The mapping e_A corresponding to each object A is an identity.*

C5 *For each identity e of $\mathcal{A}$ there is a unique object A of $\mathcal{A}$ such that $e_A = e$.*

Since we can find a one-to-one correspondence between the collection of all objects of a category and the set of all its identities, the concept of **object** plays a secondary role and thus can be even omitted in the definition of category.

The extent of the crucial importance of the concept of category in mathematics did not appeared immediately clear to Eilenberg and MacLane. We are not meaning that the two mathematicians did not see the ingenuity of the mathematical structure they had just introduced, indeed in 1945 paper itself, we read:

> "In a metamathematical sense our theory provides general concepts applicable to all branches of abstract mathematics, and so contributes to the current trend towards uniform treatment of different mathematical disciplines. In particular, it provides opportunities for the comparison of constructions and of the isomorphisms occurring in different branches of mathematics; in this way it may occasionally suggest new results by analogy.
>
> The theory also emphasizes that, whenever new abstract objects are constructed in a specified way out of given ones, it is advisable to regard the construction of the corresponding induced mappings on these new objects as an integral part of their definition. The pursuit of this program entails a simultaneous consideration of objects

> and their mappings (in our terminology, this means the consideration not of individual objects but of categories). This emphasis on the specification of the type of mappings employed gives more insight into the degree of invariance of the various concepts involved." ([ME45], pg 236)

Nevertheless, categories were not doing any mathematical work, rather they were simply a handy language for systematizing data, however clever it might be.

> "It should be observed first that the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a functor and of a natural transformation (the latter is defined in the next chapter). The idea of a category is required only by the precept that every function should have a definite class as domain and a definite class as range, for the categories are provided as the domains and ranges of functors. Thus one could drop the category concept altogether and adopt an even more intuitive standpoint, in which a functor such as "Horn" is not defined over the category of "all" groups, but for each particular pair of groups which may be given. The standpoint would suffice for the applications, inasmuch as none of our developments will involve elaborate constructions on the categories themselves." ([ME45], pg 247)

It took a dozen years before mathematicians started elaborating constructions on categories themselves. On those occasions, the fact that the concept of category has a pivotal role in many mathematical structures became clear and questions concerning the nature of categories came to light. Among the others, also MacLane tackled those foundational problems in the late 1950s and early 1960s ([Mac61]).

It is worth mentioning the fact that, although MacLane and Eilenberg introduced functor categories, they did not explore the possibility of a category of categories, neither did they notice that categories compose in two different natural ways. Also, we can assert that they did not investigate into the various properties of categories and functors, nor did they use them systematically: for these reasons it is acceptable to claim that they did not introduce category theory as such (see e.g. [MR11], pg 7). The turning point came with the publications of the books *Foundations of Algebraic Topology* ([SE52]) and *Homological Algebra* ([CE56]), where Eilenberg, Steernrod and Cartan applied category theory to algebraic topology: diagrams became a foundamental tool in proofs, functors began to play a crucial role and categories were considered in themselves (i.e. formal contexts in which one could identify, define and develop mathematics).

1.2 Categories

Definition 1.2. *A **category** $\mathcal{C}$ consists of:*

- *a class $Ob(\mathcal{C})$ of objects;*
- *for all $A, B \in Ob(\mathcal{C})$, a collection $\mathcal{C}(A,B)$ of **maps** (also called arrows or morphisms) such that:*
  - *(a) for each $A, B, C \in Ob(\mathcal{C})$ there exists a function called **composition** given by*

$$\begin{array}{ccc} \mathcal{C}(B,C) \times \mathcal{C}(A,B) & \longrightarrow & \mathcal{A}(A,C) \\ (g,f) & \longmapsto & g \circ f \end{array}$$

  - *(b) for each $A \in Ob(\mathcal{C})$ there exists an element 1_A of $\mathcal{A}(A,A)$ called **identity** on A.*

Also, the following two axioms are satisfied:

*(i) (**associativity**) for each $f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(B,C)$, $h \in \mathcal{C}(C,D)$: $(h \circ g) \circ f = h \circ (g \circ f)$;*

*(ii) (**identity**) for each $f \in \mathcal{C}(A,B)$: $f \circ 1_A = 1_B \circ f = f$.*

Every map in a category has a definite domain and codomain: if $f \in \mathcal{C}(A, B)$, we call A the **domain** and B the **codomain** of f. Also, in the definition above we are assuming that $\mathcal{C}(A, B) \cap \mathcal{C}(A', B') = \varnothing$ unless $A = A'$ and $B = B'$, given that it is possible to intersect arbitrary pairs of classes.

Example 1.1. There is a category **Set**, whose objects are sets and whose maps are functions between sets. Composition in the category is the ordinary composition of functions, and identities are the identity maps.

Example 1.2. There is a category **Grp**, whose objects are groups and whose maps are group homomorphisms. Composition and identities are the ordinary ones.

Example 1.3. There is a category **Ring**, whose objects are rings and whose maps are ring homomorphisms. Composition and identities are the ordinary ones.

Example 1.4. There is a category $\mathbf{Vect}_k$, whose objects are vector spaces over k and whose maps are linear maps between them. Composition and identities are the ordinary ones.

Example 1.5. There is a category **Top**, whose objects are topological spaces and whose maps are continuous maps between them. Composition and identities are the ordinary ones.

Example 1.6. Let's consider some language $\mathcal{L}$. There is a category **Der**, whose objects are first order formulas in the language $\mathcal{L}$. We say that there is a map from φ to ψ if there is a derivation in the classical natural deduction system with φ as the only uncancelled assumption(s) and ψ as the root of the derivation tree. Given maps $\pi : \varphi \longrightarrow \psi$ and $\pi' : \psi \longrightarrow \chi$, which means that π is a derivation tree with φ as the only uncancelled assumption(s) and ψ as the root and π' is a derivation tree with ψ as the only uncancelled assumption(s) and χ as the root, then their composition is obtained by placing the derivation tree π above every uncancelled assumption ψ in the derivation tree π'. The identity of a first order formula φ in the language $\mathcal{L}$ is given by the derivation tree of depth 0 with φ both as the only uncancelled assumption and as the root.

Example 1.7. A monoid $(M, \cdot)$ is a set equipped with an associative binary operation and a two-sided unit element. It can be represented as a category with one only object. Indeed, there is a one-to-one correspondence between monoids and one-object categories, sending every element of the monoid in a map from the only object into itself and sending the identity of the monoid into the only identity map in the category, and such that the operation $\cdot$ between elements of the monoid corresponds to the composition of maps in the correspondent one-object category.

Definition 1.3. *A map $f : A \longrightarrow B$ in a category $\mathcal{C}$ is an **isomorphism** if there exists a map $g : B \longrightarrow A$ in $\mathcal{C}$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. If this is the case we write $A \cong B$.*

Example 1.8. The isomorphisms in **Set** are bijections.

Example 1.9. The isomorphisms in **Grp** and **Ring** are group and ring isomorphisms, respectively.

Example 1.10. The isomorphisms in **Top** are the homeomorphisms. We notice that, in contrast to the situation in the previuos examples, a bijective map in **Top** is not necessarily an isomoprhism: as a counter example it suffices to consider the map:

$$\begin{array}{ccc} [0,1) & \longrightarrow & \{z \in \mathbb{C} \| z| = 1\} \\ t & \longmapsto & e^{2\pi i t} \end{array}$$

Example 1.11. A group $(G, \cdot)$ is a monoid in which every element is invertible. Exactly as in the previous example in which a monoid was seen as a one-object category, we can regard every group as a one-object category in which every map in an isomorphism.

Given some categories, it is possible to build new categories, basically in two different ways. Indeed we can define the opposite category and the product category as follows.

Definition 1.4. *Given a category $\mathcal{C}$, the **opposite category** (or dual category) $\mathcal{C}^{op}$ is described by:*

$$Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$$

and

$$\mathcal{C}^{op}(A, B) = \mathcal{C}(B, A) \text{ for each } A, B \in Ob(\mathcal{C}).$$

Due to the existence of the opposite category for any category we can consider, we can reverse every result (which means every theorem, law, observation or property) and we can find a new one, which is the dual of the previous one.

Definition 1.5. *Given two categories $\mathcal{C}$ and $\mathcal{D}$, the **product category** $\mathcal{C} \times \mathcal{D}$ is described by:*

$$Ob(\mathcal{C} \times \mathcal{D}) = Ob(\mathcal{C}) \times Ob(\mathcal{D})$$

and

$$(\mathcal{C} \times \mathcal{D})((A,B),(A',B')) = \mathcal{C}(A,A') \times \mathcal{D}(B,B') \text{ for each } (A,B),(A',B') \in Ob(\mathcal{C} \times \mathcal{D}).$$

In the definition of category, the very general word "class" is used, in order to indicate the collection of all the objects of an arbitrary category. It is well-known, though, that in mathematics the distinction between small (the collections of elements that give rise to sets) and large classes (the collections of elements that cannot be regarded as sets) is crucial: some results that holds for sets cannot be applied while working with large classes and viceversa. For this reason, we introduce a distinction between categories according to the size of their collections of objects.

Definition 1.6. *A category $\mathcal{C}$ is said to be a **small** category if the class of all its maps is small, otherwise it is said to be a **large** category. It is called a **locally small** category if for each $C, C' \in Ob(\mathcal{C})$ the class $\mathcal{C}(C,C')$ is small. It is called an **essentially small** category if it is equivalent to some small category*[1]*.*

It follows that a category is small if and only if it is locally small and its class of objects is small.

Example 1.12. The category **Set** is locally small, because for any two sets X and Y the functions from X to Y form a set. However, it is not essentially small, and in order to prove this it suffices to show that the class of isomorphism classes of sets is large. This means that if I is a set and $(A_i)_{i \in I}$ is a family of sets, then we have to show that there exists a set not isomorphic to any of the sets A_i. It is sufficient to consider

$$A := \wp(\sum_{i \in I} A_i)$$

and for each $j \in I$ the inclusion $A_j \longrightarrow \sum_{i \in I} A_i$. It follows that:

$$|A_j| \leq |\sum_{i \in I} A_i| < |A|$$

hence $|A_j| < |A|$ and in particular $A_j \ncong A$.
Similarly, **Grp**, **Ab**, **Ring** and **Top** are locally but not essentially small.

1.3 Functors

The definition of functor formalizes the notion of map between categories.

Definition 1.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **(covariant) functor** $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of:*

- *a function $Ob(\mathcal{C}) \longrightarrow Ob(\mathcal{D})$, sending any object $C \in Ob(\mathcal{C})$ into an object $F(C) \in Ob(\mathcal{D})$;*
- *for each $C, C' \in Ob(\mathcal{C})$, a function $\mathcal{C}(C,C') \longrightarrow \mathcal{D}(F(C),F(C'))$, $f \longmapsto F(f)$, such that:*
  - *(i) $F(f' \circ f) = F(f') \circ F(f)$ for every $f \in \mathcal{C}(C,C')$ and $f' \in \mathcal{C}(C',C'')$;*
  - *(ii) $F(1_C) = 1_{F(C)}$ for every $C \in Ob(\mathcal{C})$.*

*A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a (covariant) functor $\mathcal{C}^{op} \longrightarrow \mathcal{D}$.*

[1] The meaning of equivalence between categories is made clear in Definition (1.13).

We can now consider a category in which the objects are categories themselves and the maps between them are functors. This is indeed a category, and we call it **CAT**. We are not going to use this category very often, since most of the times we prefer not to work with really big collections of objects. To this purpose, we introduce the category of small categories and funtors between them and we denote it **Cat**.

Among all examples of functors, two kinds of them play a major role in category theory: forgetful and free functors.

Example 1.13. The term **forgetful functor** is informal, that is it has no precise definition. It indicates a general way of defining particular types of functors. Here are some examples.

(a) There is a functor $U : \mathbf{Grp} \longrightarrow \mathbf{Set}$ sending a group $G \in Ob(\mathbf{Grp})$ into the set $U(G)$ of its elements and every group homomorphism f into the function f itself. In other words, U forgets about the group structure of groups and group homomorphisms. Similarly we can define a forgetful functor $U : \mathbf{Ring} \longrightarrow \mathbf{Set}$ forgetting about the ring structure of rings and ring homomorphisms, as well as a forgetful functor $U : \mathbf{Vect}_k \longrightarrow \mathbf{Set}$ forgetting about the vector space structure of vector spaces and about linarity of linear maps.

(b) Forgetful functors need not forget about the whole structure of a category. For instance there is a functor $U : \mathbf{Ring} \longrightarrow \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of all abelian groups, forgetting about the multiplicative structure of a ring but remembering the underlying additive group. Similarly, if $\mathbf{Mon}$ is the category of monoids and monoid homomorphisms, there is a functor $U : \mathbf{Ring} \longrightarrow \mathbf{Mon}$ that forgets the additive structure and remembers the underlying multiplicative monoid. In addition to that, we can also consider the functor $U : \mathbf{Ab} \longrightarrow \mathbf{Grp}$ that forgets about the commutativity of abelian groups.

Example 1.14. We can also consider the **free functors**, which are in some sense dual to the forgetful functors (to be precise, there is an adjunction between them). There is a functor $F : \mathbf{Set} \longrightarrow \mathbf{Grp}$, sending a set S into the free group $F(S)$ on S and every function $f : S \longrightarrow S'$ in $\mathbf{Set}$ into the group homomorphism $F(f) : F(S) \longrightarrow F(S')$[2]. Similarly, there is a functor $F : \mathbf{Set} \longrightarrow \mathbf{CRing}$, where $\mathbf{CRing}$ is the category of commutative ring and ring homomorphims between them, sending every set S into the ring of polynomials over $\mathbb{Z}$ in commutative variables $\{x_s\}_{s\in S}$. Also, there is a functor $F : \mathbf{Set} \longrightarrow \mathbf{Vect}_k$, sendind very set S into a vector space with basis S (any two such vector spaces are isomorphic).

Example 1.15. Let G be a group regarded as a one-object category $\mathcal{G}$. A functor $A : \mathcal{G} \longrightarrow \mathbf{Set}$ assigns to the unique object $\mathbb{G} \in \mathcal{G}$ a set S and to every map g in $\mathcal{G}$ a function $A(g) : S \longrightarrow S$, satisfying the conditions in the definition of functor. If we write $(F(g))(s) = g \cdot s$, we notice that the functor A amounts to a set S together with a function

$$\begin{array}{ccc} G \times S & \longrightarrow & S \\ (g, s) & \longmapsto & g \cdot s \end{array}$$

such that $(gg') \cdot s = g' \cdot (g \cdot s)$ and $1_G \cdot s = s$, for every $g, g' \in G$, $s \in S$. Therefore, the functor A is a set equipped with a left action by G.

Example 1.16. Let's fix a vector space W over the field k. For any k-vector space V we can define the vector space $\mathbf{Hom}(V, W)$ of the linear maps from V to W where the operations are defined pointwise, and for any linear map $f : V \longrightarrow V'$ we can define the linear map $\mathbf{Hom}(f, W) : \mathbf{Hom}(V', W) \longrightarrow \mathbf{Hom}(V, W)$ by $\mathbf{Hom}(f, W)(q) := q \circ f$ for every $q \in \mathbf{Hom}(V', W)$. This defines a contravariant funtor $\mathbf{Hom}(-, W) : \mathbf{Vect}_k^{op} \longrightarrow \mathbf{Vect}_k$. Similarly, if we consider the field k in place of the vector space W, we can define a contravariant functor $(-)^* = \mathbf{Hom}(-, k) : \mathbf{Vect}_k^{op} \longrightarrow \mathbf{Vect}_k$, which turns out to send each k-vector space into its dual.

[2]More precisely, we can think of the elements of $F(S)$ as formal expressions or words built from the elements of S. For instance, the word $yx^{-4}z^3y^{-1}$ is obtained from the elements x, y, z of some set. Furthermore, any map between sets gives rise to a group homomorphism. For instance, if we take the map $f : \{x, y, z\} \longrightarrow \{u, v\}$ defined by $f(x) = f(y) = u$ and $f(z) = v$, this gives rise to a group homomorphism $F(f) : F(\{x, y, z\}) \longrightarrow F(\{u, v\})$ which for example sends the word $yx^{-4}z^3y^{-1} \in F(\{x, y, z\})$ to $uu^{-4}v^3u^{-1} = u^{-3}v^3u^{-1} \in F(\{u, v\})$.

Definition 1.8. *A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is **faithful** [**full**] if for every $C, C' \in Ob(\mathcal{C})$ the function*

$$\begin{array}{ccc} \mathcal{C}(C, C') & \longrightarrow & \mathcal{D}(F(C), F(C')) \\ f & \longmapsto & F(f) \end{array}$$

is injective [surjective].

In other words, a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is faithful if and only if for each $C, C' \in Ob(\mathcal{C})$ and $g \in \mathcal{D}(F(C), F(C'))$, there is *at most* one arrow $C \longrightarrow C'$ in $\mathcal{C}$ that sends F into g, and it is full if for each $C, C' \in Ob(\mathcal{C})$ and $g \in \mathcal{D}(F(C), F(C'))$, there is *at least* one arrow $C \longrightarrow C'$ in $\mathcal{C}$ that sends F into g.

Definition 1.9. *Let $\mathcal{C}$ be a category. A **subcategory** $\mathcal{S}$ of $\mathcal{C}$ consists of:*

- *a subclass $Ob(\mathcal{S})$ of $Ob(\mathcal{C})$;*
- *for each $S, S' \in Ob(\mathcal{S})$ there is a subclass $\mathcal{S}(S, S')$ of $\mathcal{C}$ such that it is closed under composition and identities.*

*$\mathcal{S}$ is said to be a **full subcategory** if for all $S, S' \in Ob(\mathcal{S})$ it happens that $\mathcal{S}(S, S') = \mathcal{C}(S, S')$. Whenever $\mathcal{S}$ is a subcategory of $\mathcal{C}$ we can find an **inclusion functor**:*

$$\begin{array}{cccc} I : & \mathcal{S} & \longrightarrow & \mathcal{C} \\ & S & \longmapsto & S \\ & f : S \longrightarrow S' & \longmapsto & f : S \longrightarrow S' \end{array}$$

Trivially, the inclusion functor I is faithful and it is full if and only if $\mathcal{S}$ is a full subcategory. Generally speacking, the image of a functor is not a subcategory.

1.4 Natural transformations

Other than the notion of map between categories, which is a functor, we can also take into account the notion of map between functors. For this reason we introduce the following definition.

Definition 1.10. *Let $\mathcal{C}$, $\mathcal{D}$ be two categories and $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be two functors between them. A **natural transformation** $\alpha : F \longrightarrow G$ is a family $(\alpha_C : F(C) \longrightarrow G(C))_{C \in Ob(\mathcal{C})}$ of maps in $\mathcal{D}$ such that for every $f : C \longrightarrow C'$ in $\mathcal{C}$ the following square commutes:*

$$\begin{tikzcd} F(C) \arrow[r, "F(f)"] \arrow[d, "\alpha_C"'] & F(C') \arrow[d, "\alpha_{C'}"] \\ G(C) \arrow[r, "G(f)"'] & G(C') \end{tikzcd} \tag{1.1}$$

We write $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$ *to mean that α is a natural trasformation from F to G.*

Example 1.17. Given a commutative ring R, the matrices $M_n(R)$ over the ring R form a monoid, with the matrix multiplication as operation. Given a ring homomorphism $R \longrightarrow S$, then there is a monoid homomorphism $M_n(R) \longrightarrow M_n(S)$. Hence, we can define a functor $M_n : \mathbf{CRing} \longrightarrow \mathbf{Mon}$. Also, we can consider the forgetful functor that forgets the additive structure of a commutative ring, giving rise to a monoid: $U : \mathbf{CRing} \longrightarrow \mathbf{Mon}$.
We can consider the natural transformation

$$\begin{tikzcd} \mathbf{CRing} \arrow[r, bend left, "M_n"] \arrow[r, bend right, "U"'] \arrow[r, phantom, "\Downarrow det"] & \mathbf{Mon} \end{tikzcd}$$

defined by the family

$$(M_n(R) \xrightarrow{det_R} U(R))_{R \in Ob(\mathbf{CRing})}.$$

Indeed, from the property of the determinant $det_R(XY) = det_R(X)det_R(Y)$ and $det_R(\mathbb{I}) = 1$, we infer that det_R is a monoid morphism, thus a map in **Mon**, for every ring R. In addition, for every $f \in \mathbf{CRing}(R, S)$ the following square commutes:

$$\begin{tikzcd} M_n(R) \arrow[r, "\mathbf{CRing}(f)"] \arrow[d, "det_R"'] & M_n(S) \arrow[d, "det_S"] \\ U(R) \arrow[r, "U(f)"'] & U(S) \end{tikzcd}$$

since for every $X \in M_n(R)$: $f(det_R(X)) = det_S(f(X))$.

We can easily notice that we can compose natural trasformations. For instance, let's consider the natural trasformations α and β such that:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{D} \end{tikzcd}$$

Then there is a composite natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \mathcal{D} \end{tikzcd}$$

defined by $(\beta \circ \alpha)_C = \beta_C \circ \alpha_C$, for all $C \in \mathcal{C}$. In addition, we can also consider the identity natural transformation $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow 1_F"] & \mathcal{D} \end{tikzcd}$. Therefore, the natural transformations admit composition and identities. The following definition makes sense.

Definition 1.11. *Let $\mathcal{C}$, $\mathcal{D}$ be two categories. The* ***functor category*** *$\mathcal{D}^{\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{D}$ is the category whose objects are the functors from $\mathcal{C}$ to $\mathcal{D}$ and whose maps are the natural transformations between them.*

Given two functors, we do not often care at all whether the functors are equal. In fact, we are more interested in their structures, and in the way those reflect their jobs. As a consequence, we introduce the notion of natural isomorphism.

Definition 1.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, F and G functors and α a natural trasformation* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$ *. α is said to be a* ***natural isomorphism*** *if equivalently:*

(i) α is an isomorphism in the functor category $\mathcal{D}^{\mathcal{C}}$;

(ii) $\alpha_C : F(C) \longrightarrow G(C)$ is an isomorphism for all $C \in Ob(\mathcal{C})$.

If this is the case we write $F \cong G$, and we say that $F(C) \cong G(C)$ ***naturally in*** *C for every $C \in Ob(\mathcal{C})$.*

Lemma 1.1. *The two definitions (i) and (ii) of natural isomorphism are equivalent.*

Proof. $(i) \longrightarrow (ii)$ Let $\alpha : F \longrightarrow G$ be a natural transformation from $\mathcal{C}$ to $\mathcal{D}$ and assume it to be an isomorphism in the category $\mathcal{D}^{\mathcal{C}}$. Hence, there is $\beta : G \longrightarrow F$ in $\mathcal{D}^{\mathcal{C}}$ such that $\beta \circ \alpha = 1_F$ and $\alpha \circ \beta = 1_G$. This means that $\beta_C \circ \alpha_C = 1_{F(C)}$ and $\alpha_C \circ \beta_C = 1_{G(C)}$, for all $C \in \mathcal{C}$. So, α_C is an isomorphism for every $C \in \mathcal{C}$.

$(ii) \longrightarrow (i)$ Let $\alpha : F \longrightarrow G$ be a natural transformation from $\mathcal{C}$ to $\mathcal{D}$ and assume that $\alpha_C : F(C) \longrightarrow G(C)$ is an isomorphism for all $C \in \mathcal{C}$. Then for each $C \in \mathcal{C}$ there is some $\beta_C : G(C) \longrightarrow F(C)$ such that $\beta_C \circ \alpha_C = 1_{F(C)}$ and $\alpha_C \circ \beta_C = 1_{G(C)}$. Now the family of maps in $\mathcal{D}$ given by $(\beta_C)_{C \in Ob(\mathcal{C})}$ can be regarded as a natural transformation β, so as a map in $\mathcal{D}^{\mathcal{C}}$. Therefore we have found $\beta \in (\mathcal{D}^{\mathcal{C}})(G, F)$ such that $\beta \circ \alpha = 1_F$ and $\alpha \circ \beta = 1_G$, which means that α is an isomorphism in $\mathcal{D}^{\mathcal{C}}$. $\square$

Example 1.18. Given a field k, let $\mathbf{FDVect}_k$ be the category of finite dimensional k-vector spaces and linear maps between them. There is a contravariant functor

$$(-)^* = \mathbf{Hom}(-, k) : \mathbf{FDVect}_k \longrightarrow \mathbf{FDVect}_k$$

defining the dual vector space construction. Similarly, we can define a covariant functor

$$(-)^{**} : \mathbf{FDVect}_k \longrightarrow \mathbf{FDVect}_k$$

sending every vector space into its dual. Moreover there is a canonical isomorphism $\alpha_V : V \longrightarrow V^{**}$ for every $V \in Ob(\mathbf{FDVect}_k)$.
Having said that, we can construct a natural isomorphism

$$\begin{tikzcd} \mathbf{FDVect}_k \arrow[r, bend left=50, "1_{\mathbf{FDVect}_k}"{name=U}] \arrow[r, bend right=50, "(-)^{**}"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

thus $V \cong V^{**}$ naturally in V.

Definition 1.13. *An* ***equivalence*** $(F, G, \eta, \varepsilon)$ *between two categories* $\mathcal{C}$ *and* $\mathcal{D}$ *consists of a pair of functors*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

and a pair of natural isomorphisms

$$\eta : 1_{\mathcal{C}} \longrightarrow G \circ F,\ \varepsilon : F \circ G \longrightarrow 1_{\mathcal{D}}.$$

If this is the case, we say that $\mathcal{C}$ *and* $\mathcal{D}$ *are* ***equivalent*** *and that the functors* F *and* G *are* ***equivalences****. We write* $\mathcal{C} \simeq \mathcal{D}$.

Definition 1.14. *A functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *is* ***essentially surjective on objects*** *if for all* $D \in \mathcal{D}$ *there exists* $C \in Ob(\mathcal{C})$ *such that* $F(C) \cong D$.

Proposition 1.1. *A functor is an equivalence if and only if it is full, faithful and essentially surjective on objects.*

Proof. $(\Rightarrow)$ Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an equivalence, hence there exists a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that $\eta : 1_{\mathcal{C}} \xrightarrow{\sim} G \circ F$ and $\varepsilon : F \circ G \xrightarrow{\sim} 1_{\mathcal{D}}$.
F is faithful. Since η is an isomorphism, $\eta_C : C \xrightarrow{\sim} GF(C)$ for any $C \in \mathcal{C}$, and for any $f \in \mathcal{C}(C, C')$ the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\eta_C"'] & C' \arrow[d, "\eta_{C'}"] \\ GF(C) \arrow[r, "GF(f)"'] & GF(C') \end{tikzcd}$$

Thus, if $F(f) = F(f')$ then applying G one gets $GF(f) = GF(f')$ and so, by the commutativity of the previous diagram and the fact that η_C and $\eta_{C'}$ are isomorphisms, $f = f'$. So F is faithful. Similarly also G is.
F is full. Let $g \in \mathcal{D}(F(C), F(C'))$. We consider:

$$\begin{tikzcd} C \arrow[d, "\eta_C"'] & C' \arrow[d, "\eta_{C'}"] \\ GF(C) \arrow[r, "G(g)"'] & GF(C') \end{tikzcd}$$

Being $\eta_{C'}$ an isomorphism, we can define $f := (\eta_{C'})^{-1} \circ G(g) \circ \eta_C$. Also, by the commutativity of:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\eta_C"'] & C' \arrow[d, "\eta_{C'}"] \\ GF(C) \arrow[r, "G(g)"'] & GF(C') \end{tikzcd}$$

we infer that $GF(f) = G(g)$. But G is faithful, hence $F(f) = g$, thus F is full.
F is essentially surjective on objects. Let $D \in Ob(\mathcal{D})$. We know from $\varepsilon : F \circ G \xrightarrow{\sim} 1_{\mathcal{D}}$ that $\varepsilon_D : D \longrightarrow FG(D)$ is an isomorphism so $D \cong FG(D)$, and $G(D) \in Ob(\mathcal{C})$. Therefore F is essentially surjective on objects.

($\Leftarrow$) Suppose F faithful, full and essentially surjective on objects. Take an arbitrary $D \in Ob(\mathcal{D})$. By essential surjectivity on objects, there exists an object of $\mathcal{C}$ such that its image through F is isomorphic to D. Take $G(D) \in Ob(\mathcal{C})$ to be this object and consider $\varepsilon_D : FG(D) \xrightarrow{\sim} D$. Let $g \in \mathcal{D}(D, D')$ and consider:

$$\begin{tikzcd} FG(D) \arrow[d, "\varepsilon_D"'] & FG(D') \arrow[d, "\varepsilon_{D'}"] \\ D \arrow[r, "g"'] & D' \end{tikzcd}$$

so one gets $(\varepsilon_{D'})^{-1} \circ g \circ \varepsilon_D : FG(D) \longrightarrow FG(D')$. But F is faithful and full, so for every $D, D' \in Ob(\mathcal{D})$ there exists unique $G(g) : G(D) \longrightarrow G(D')$ (more formally: for every couple of elements D, D' there are unique C, C' such that $F(C) = D$ and $F(C') = D'$ and such that there is a morphism between them; so we define G on g as we did). Also, by the previous diagram, we deduce that $FG(g)$ must be exactly $(\varepsilon_{D'})^{-1} \circ g \circ \varepsilon_D$. Defined in such a way, G is clearly a functor from $\mathcal{D}$ to $\mathcal{C}$ and $\varepsilon : F \circ G \longrightarrow 1_{\mathcal{D}}$ defined on every D as before gives clearly rise to a natural isomorphism.
Let's take now $C \in Ob(\mathcal{C})$. Then $F(C) \in Ob(\mathcal{D})$ and we can consider, given what has just been said, $\varepsilon_{F(C)} : F(C) \xrightarrow{\sim} FGF(C)$. Since F is faithful and full, we can well-define $F^{-1}(\varepsilon_{F(C)}) : C \xrightarrow{\sim} GF(C)$. This gives rise to a natural isomorphism (since ε is) $\eta : 1_{\mathcal{C}} \longrightarrow G \circ F$. □

Example 1.19. Let $\mathcal{M}$ be the full subcategory of **Cat** whose objects are the small one-object categories. By the definition of category, the maps from an object of a small one-object category to itself form a monoid under composition. So, there is a functor $F : \mathcal{M} \longrightarrow$ **Mon** sending a one-object category to the monoid of maps from the single object to itself. F is full, faithful and surjective on objects. Therefore $\mathcal{M} \cong$ **Mon**.

Corollary 1.1. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a full and faithful functor. Then $\mathcal{C}$ is equivalent to the full subcategory $\mathcal{C}'$ of $\mathcal{D}$ whose objects are those of the form $F(C)$ for some $C \in Ob(\mathcal{C})$.*

Proof. Let's consider the functor $F' : \mathcal{C} \longrightarrow \mathcal{C}'$ defined by $F'(C) := F(C)$ for every $C \in Ob(\mathcal{C})$. It is full and faithful, since F is. Besides, it is essentially surjective on objects, by definition of $\mathcal{C}'$. By the previous proposition we can conclude. □

Chapter 2

Adjoint functors

An adjunction between two functors can be defined in many different ways. We want now to present different definitions of adjunction between two categories (which means that two functors are one the adjoint of the other), in order to prove then their being equivalent.

The notion of adjunction, as we will see, is the real turning point that makes possible a categorical treatment of logic. In fact, the category-theoretic approach to logic consists in recognizing the hidden essentially adjoint-functorial behaviour at the core of logical connectives and quantifiers. Though this great intuition came first to Lawvere's mind, he was not the first one to realize the pivotal importance of adjoint functors. The notion of adjoint functor was first introduced in 1956 by Kan. He soon realized that adjunction was the key to unify various results that he had obtained in the previous years. Before publishing the unified version of those results in 1958, he had to write a paper on adjoint functors only and on that occasion he realized how truly general the notion of adjunction is.

Definition 2.1 (Adjunction via isomorphism between categories)**.** *Let* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *be categories and functors.* F *is* ***left adjoint*** *to* G *and* G *is* ***right adjoint*** *to* F *(we write* $F \dashv G$*) if*

$$\mathcal{D}(F(C), D) \cong \mathcal{C}(C, G(D)) \tag{2.1}$$

naturally in $C \in Ob(\mathcal{C})$ *and* $D \in \mathcal{D}$*. More precisely, there exists an isomorphism*

$$\begin{array}{ccc} \mathcal{D}(F(C), D) & \longrightarrow & \mathcal{C}(C, G(D)) \\ (F(C) \xrightarrow{g} D) & \mapsto & (C \xrightarrow{\overline{g}} G(D)) \\ (F(C) \xrightarrow{\overline{f}} D) & \leftarrow\!\shortmid & (C \xrightarrow{f} G(D)) \end{array}$$

(where $\overline{f}$ *is called the* ***transpose*** *of* f*) such that the following two axioms are satisfied:*

$$\overline{(F(C) \xrightarrow{g} D \xrightarrow{q} D')} = (C \xrightarrow{\overline{g}} G(D) \xrightarrow{G(q)} G(D')) \tag{2.2}$$

$$\overline{(C' \xrightarrow{p} C \xrightarrow{f} G(D))} = (F(C') \xrightarrow{F(p)} F(C) \xrightarrow{\overline{f}} D) \tag{2.3}$$

for every g*,* q*,* p *and* f*.*
An ***adjunction*** *between* $\mathcal{C}$ *and* $\mathcal{D}$ *is a triple* $(F, G, \bar{-})$ *satisfying (2.1), (2.2) and (2.3).*

In the conditions (2.2) and (2.3) of the definition, in makes no difference whether to put the bar over the right-hand side or the left-hand side of the equalities: the bar is self-inverse. Also, it can be easily seen that the two conditions (2.2) and (2.3) can be replaced by the unique request:

$$\overline{(C' \xrightarrow{p} C \xrightarrow{f} G(D) \xrightarrow{G(q)} G(D'))} = (F(C') \xrightarrow{F(p)} F(C) \xrightarrow{\overline{f}} D \xrightarrow{q} D')$$

for every q, p and f.

Definition 2.2 (Adjunction via unit and counit)**.** *Let* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *be categories and functors. F is **left adjoint** to G and G is **right adjoint** to F (we write $F \dashv G$) if there exist two natural transformations:*

$$\eta : 1_{\mathcal{C}} \longrightarrow G \circ F \text{ and } \varepsilon : F \circ G \longrightarrow 1_{\mathcal{D}}$$

*called **unit** and **counit** respectively, satisfying the following **triangle identities**:*

$$\begin{tikzcd} F(C) \arrow[r, "F(\eta_C)"] \arrow[dr, "1_{F(C)}"'] & FGF(C) \arrow[d, "\varepsilon_{F(C)}"] & G(D) \arrow[r, "\eta_{G(D)}"] \arrow[dr, "1_{G(D)}"'] & GFG(D) \arrow[d, "G(\varepsilon_D)"] \\ & F(C) & & G(D) \end{tikzcd} \tag{2.4}$$

*An **adjunction** between $\mathcal{C}$ and $\mathcal{D}$ is a quadruplet $(F, G, \eta, \varepsilon)$ where* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *are functors and $\eta : 1_{\mathcal{C}} \longrightarrow G \circ F$, $\varepsilon : F \circ G \longrightarrow 1_{\mathcal{D}}$ are natural transformations satisfying (2.4).*

Definition 2.3. *Let $\mathcal{C}$ be a category. An object $I \in Ob(\mathcal{C})$ is **initial** if for every $C \in Ob(\mathcal{C})$ there is exactly one map $I \longrightarrow C$. An object $T \in Ob(\mathcal{C})$ is **terminal** if for every $C \in Ob(\mathcal{C})$, there is exactly one map $C \longrightarrow T$.*

It can be easily seen that initial objects, as well as terminal objects (one concept is dual to the other), are unique up to isomorphism, which means that if I, I' are both initial objects in some category, then there is a unique isomorphism between them. We call $\mathbf{1}$ the terminal object of the category **CAT**, containing only one object and only the identity map of this object.

Definition 2.4. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be categories and $P : \mathcal{A} \longrightarrow \mathcal{C}$, $Q : \mathcal{B} \longrightarrow \mathcal{C}$ be functors. The **comma category** $(P \Rightarrow Q)$ is the category where objects are triples (A, h, B) with $A \in Ob(\mathcal{A})$, $B \in Ob(\mathcal{B})$, $h \in \mathcal{C}(P(A), Q(B))$ and maps $(A, h, B) \longrightarrow (A', h', B')$ are pairs $(f : A \longrightarrow A', g : B \longrightarrow B')$ of maps sich that the following square commutes:*

$$\begin{tikzcd} P(A) \arrow[r, "P(f)"] \arrow[d, "h"'] & P(A') \arrow[d, "h'"] \\ Q(B) \arrow[r, "Q(g)"'] & Q(B') \end{tikzcd}$$

Let $\mathcal{C}$ be a category. Any object $C \in Ob(\mathcal{C})$ can be described by a functor $C : \mathbf{1} \longrightarrow \mathcal{C}$, namely the object to which the unique object of $\mathbf{1}$ is mapped. Let's consider now the categories $\mathcal{C}$ and $\mathcal{D}$, the functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ and an object viewed as a functor $C : \mathbf{1} \longrightarrow \mathcal{C}$. The comma category $(C \Rightarrow G)$ has as objects the pairs $(D, f : C \longrightarrow G(D))$ and a map $(D, f) \longrightarrow (D', f')$ in $(C \Rightarrow G)$ is a map $q : D \longrightarrow D'$ in $\mathcal{D}$ such that the triangle

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "f'"'] & G(D) \arrow[d, "G(q)"] \\ & G(D') \end{tikzcd}$$

commutes. Dually, we can consider the functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ and the object viewed as a functor $D : \mathbf{1} \longrightarrow \mathcal{D}$, and we can form the comma category $(F \Rightarrow D)$. Its objects are pairs $(C, g : F(C) \longrightarrow D)$ and a map $(C, g) \longrightarrow (C', g')$ in $(F \Rightarrow D)$ is a map $p : C \longrightarrow C'$ such that the triangle

$$\begin{tikzcd} F(C) \arrow[r, "g"] \arrow[d, "F(p)"'] & D \\ F(C') \arrow[ur, "g'"'] & \end{tikzcd}$$

commutes.

Definition 2.5 (Adjunction via initial objects). *Let* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *be categories and functors. F is **left adjoint** to G and G is **right adjoint** to F (we write $F \dashv G$) if there is a natural transformation $\eta : 1_{\mathcal{C}} \longrightarrow G \circ F$ such that $\eta_C : C \longrightarrow GF(C)$ is an initial object in the comma category $(C \Rightarrow G)$ for every $C \in Ob(\mathcal{C})$.*
*An **adjunction** $(F, G, (\eta_C)_{C \in Ob(\mathcal{C})})$ between $\mathcal{C}$ and $\mathcal{D}$ is a choice for F, G functors and η_C natural transformation as above.*

In the notation of the definition (2.5) above, given $C \in Ob(\mathcal{C})$, claiming that $\eta_C : C \longrightarrow GF(C)$ is an initial object in the comma category $(C \Rightarrow G)$ means that for any object $(D, f : C \longrightarrow G(D)) \in Ob(C \Rightarrow G)$ there exists a unique map in the comma category $(C \Rightarrow G)$ from $(F(C), \eta_C)$ to (D, f). And this means that for any $(D, f) \in Ob(C \Rightarrow G)$ there exists a unique map $q : F(C) \longrightarrow D$ in $\mathcal{D}$ such that

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & GF(C) \arrow[d, "G(q)"] \\
& G(D)
\end{tikzcd}$$

commutes.

We can also have the dual statement of the definition (2.5).

Definition 2.6 (Adjunction via terminal objects). *Let* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *be categories and functors. F is **left adjoint** to G and G is **right adjoint** to F (we write $F \dashv G$) if there is a natural transformation $\varepsilon : F \circ G \longrightarrow 1_{\mathcal{D}}$ such that $\varepsilon_D : FG(D) \longrightarrow D$ is a terminal object in the comma category $(F \Rightarrow D)$ for every $D \in Ob(\mathcal{D})$.*
*An **adjunction** $(F, G, (\varepsilon_D)_{D \in Ob(\mathcal{D})})$ between $\mathcal{C}$ and $\mathcal{D}$ is a choice for F, G functors and ε_D natural transformation as above.*

In the notation of the definition (2.6) above, given $D \in Ob(\mathcal{D})$, claiming that $\varepsilon_D : FG(D) \longrightarrow D$ is a terminal object in the comma category $(F \Rightarrow D)$ means that for any object $(C, g : F(C) \longrightarrow D) \in Ob(F \Rightarrow D)$ there exists a unique map in the comma category $(F \Rightarrow D)$ from (C, g) to $(G(D), \varepsilon_D)$. And this means that for any $(C, g) \in Ob(F \Rightarrow D)$ there exists a unique map $p : C \longrightarrow G(D)$ in $\mathcal{C}$ such that

$$\begin{tikzcd}
F(C) \arrow[r, "g"] \arrow[d, "F(p)"'] & D \\
FG(D) \arrow[ur, "\varepsilon_D"']
\end{tikzcd}$$

commutes.

We decide now to take the definition (2.1) as the definition of adjunction, and we prove that the definitions (2.2) and (2.5) are equivalent to the previous one.

Theorem 2.1. *An adjunction $(F, G, \underset{\cdot}{-})$ between the categories $\mathcal{C}$ and $\mathcal{D}$ (where $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ are functors) is completely determined by each one of the following conditions:*

(i) $(F, G, \eta, \varepsilon)$ is a quadruplet where F, G are functors and $\eta : 1_{\mathcal{C}} \longrightarrow G \circ F$, $\varepsilon : F \circ G \longrightarrow 1_{\mathcal{D}}$ (called unit and counit) are natural transformations satisfying the triangle identities (2.4).

(ii) $(F, G, (\eta_C)_{C \in Ob(\mathcal{C})})$ is a set where $\eta : 1_{\mathcal{C}} \longrightarrow G \circ F$ is a natural transformation such that η_C is an initial object in the comma category $(C \Rightarrow G)$ for every $C \in Ob(\mathcal{C})$.

(iii) $(G, (F_0(C), \eta_C)_{C \in Ob(\mathcal{C})})$ where, given $C \in Ob(\mathcal{C})$, $F_0(C) \in Ob(\mathcal{D})$ and $\eta_C : C \longrightarrow GF_0(C)$ is a map in $\mathcal{C}$ such that for any $D \in Ob(\mathcal{D})$ and any $f : C \longrightarrow G(D)$ in $\mathcal{C}$ there exists a unique

$q : F_0(C) \longrightarrow D$ such that the triangle

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & GF_0(C) \arrow[d, "G(q)"] \\ & G(D) \end{tikzcd}$$

commutes.

(iv) $(F, G, (\varepsilon_D)_{D \in Ob(\mathcal{D})})$ is a set where $\varepsilon : F \circ G \longrightarrow 1_{\mathcal{D}}$ is a natural transformation such that ε_D is a terminal object in the comma category $(F \Rightarrow D)$ for every $D \in Ob(\mathcal{D})$.

(v) $(F, (G_0(D), \varepsilon_D)_{D \in Ob(\mathcal{D})})$ where, given $D \in Ob(\mathcal{D})$, $G_0(D) \in Ob(\mathcal{C})$ and $\varepsilon_D : FG_0(D) \longrightarrow D$ is a map in $\mathcal{D}$ such that for any $C \in Ob(\mathcal{C})$ and any $g : F(C) \longrightarrow D$ in $\mathcal{D}$ there exists a unique $p : C \longrightarrow G_0(D)$ such that the triangle

$$\begin{tikzcd} F(C) \arrow[r, "g"] \arrow[d, "F(p)"'] & D \\ FG_0(D) \arrow[ur, "\varepsilon_D"'] & \end{tikzcd}$$

commutes.

Proof. The statements (iii) and (v) are straightforward consequences of (ii) and (iv) respectively. Also, (iv) is obtained from (ii) by duality. Consequently, we will simply show that (i) is equivalent to the definition (2.1) and to (ii).

(i). Let $(F, G, \bar{\cdot})$ be an adjunction, where $f \dashv G$ and $\bar{\cdot}$ is the transposition

$$\bar{\cdot} : \mathcal{D}(F(C), D) \xrightarrow{\cong} \mathcal{C}(C, G(D))$$

Let $C \in Ob(\mathcal{C})$, hence $F(C) \in Ob(\mathcal{D})$ and $1_{F(C)} : F(C) \longrightarrow F(C)$ is a map in $\mathcal{D}$ by definition of category. By (2.2), there is a map $\eta_C := \overline{1_{F(C)}} : C \longrightarrow GF(C)$ in $\mathcal{D}$. Hence, by the arbitrariness of $C \in Ob(\mathcal{C})$, we can define the unit

$$\eta : 1_{\mathcal{C}} \longrightarrow G \circ F.$$

This is a natural transformation, indeed for every $p : C \longrightarrow C'$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "p"] \arrow[d, "\eta_C"'] & C' \arrow[d, "\eta_{C'}"] \\ GF(C) \arrow[r, "GF(p)"'] & GF(C') \end{tikzcd}$$

since by (2.2) and (2.3):

$$\eta_{C'} \circ p = \overline{1_{C'}} \circ p = \overline{1_{C'} \circ F(p)} = \overline{F(p) \circ 1_{F(C')}} = GF(p) \circ \overline{1_{F(C')}} = GF(p) \circ \eta_C.$$

Similarly we can define the counit

$$\varepsilon : F \circ G \longrightarrow 1_{\mathcal{D}}$$

which is a natural trasformation as well. In addition, the triangle identities (2.4) hold. Indeed, by (2.3) we infer

$$\overline{(C \xrightarrow{\eta_C} GF(C) \xrightarrow{1_{GF(C)}} GF(C))} = (F(C) \xrightarrow{F(\eta_C)} FGF(C) \xrightarrow{\varepsilon_{F(C)}} F(C))$$

but the left-hand side is $\overline{\eta_C} = \overline{\overline{1_{F(C)}}} = 1_{F(C)}$, so the first identity holds. The second one follows by duality.

Let's consider now a pair of natural transformations (η, ε) satisfying the triangle identities (2.4), and let's show that there is an adjunction $F \vdash G$ in the respect of definition (2.1). We define the isomorphism

$$\begin{array}{ccc} \mathcal{D}(F(C), D) & \longrightarrow & \mathcal{C}(C, G(D)) \\ (F(C) \xrightarrow{g} D) & \mapsto & (C \xrightarrow{\overline{g}} G(D)) \\ (F(C) \xrightarrow{\overline{f}} D) & \leftarrow\!\shortmid & (C \xrightarrow{f} G(D)) \end{array}$$

by:

$$\overline{g} := G(g) \circ \eta_C$$

for any $g : F(C) \longrightarrow D$ in $\mathcal{D}$ and

$$\overline{f} := \varepsilon_D \circ F(f)$$

for any $f : C \longrightarrow G(D)$ in $\mathcal{C}$. This is indeed an isomorphism, as $g \longmapsto \overline{g}$ and $f \longmapsto \overline{f}$ are mutually inverse: given a map $g : F(C) \longrightarrow D$ in $\mathcal{D}$ the following diagram commutes:

$$\begin{tikzcd} F(C) \arrow[r, "F(\eta_C)"] \arrow[dr, "1_{F(C)}"'] & FGF(C) \arrow[r, "FG(g)"] \arrow[d, "\varepsilon_{F(C)}"] & FG(D) \arrow[d, "\varepsilon_D"] \\ & F(C) \arrow[r, "g"'] & D \end{tikzcd}$$

thus

$$\varepsilon_D \circ FG(g) \circ F(\eta_C) = g \circ 1_{F(C)}$$

$$\varepsilon_D \circ F(G(g) \circ \eta_C) = g$$

$$\varepsilon_D \circ F(\overline{g}) = g$$

$$\overline{\overline{g}} = g$$

and dually $\overline{\overline{f}} = f$ for any $f : C \longrightarrow G(D)$ in $\mathcal{C}$. Finally, given $g : F(C) \longrightarrow D$ and $q : D \longrightarrow D'$ in $\mathcal{D}$, from the way the isomorphism $\overline{\cdot}$ was defined:

$$\overline{q \circ g} = G(q \circ g) \circ \eta_C = G(q) \circ G(g) \circ \eta_C = G(q) \circ \overline{g}$$

which gives exactly the condition (2.2). By duality, also (2.3) holds.

(ii). Let $(F, G, \eta, \varepsilon)$ be an adjunction between the categories $\mathcal{C}$ and $\mathcal{D}$ with unit η and counit ε, satisfying the triangle identities (2.4). Let C be an object of $\mathcal{C}$. Then the unit map $\eta_C : C \longrightarrow GF(C)$ is an initial object in the comma category $(C \Rightarrow G)$, which means that for any object $D \in Ob(\mathcal{D})$ and any map $f : C \longrightarrow G(D)$ in $\mathcal{C}$ there is a unique map $q : F(C) \longrightarrow D$ such that

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & GF(C) \arrow[d, "G(q)"] \\ & G(D) \end{tikzcd}$$

commutes. Indeed, by the previous point of the proof we have that $G(q) \circ \eta_C = \overline{q}$, so the previous triangle commutes if and only if $f = \overline{q}$ if and only if $q = \overline{f}$, and so $\overline{f}$ is the unique map we were searching for.

Let's consider now the functors F and G and $\varepsilon : F \circ G \longrightarrow 1_{\mathcal{D}}$ a natural transformation such that ε_D is a terminal object in the comma category $(F \Rightarrow D)$ for every $D \in Ob(\mathcal{D})$. We need to show that there exists a unique natural transformation $\varepsilon F \circ G \longrightarrow 1_{\mathcal{D}}$ such that the pair (η, ε) satisfies the triangle identities (2.4). Let $D \in Ob(\mathcal{D})$ be fixed. We define ε_D to be the map

$(FG(D), \eta_{G(D)}) \longrightarrow (D, 1_{G(D)})$ in the comma category $(G(D) \Rightarrow G)$.
Naturality. Let $q : D \longrightarrow D'$ in $\mathcal{D}$. The following diagrams commute:

$$\begin{tikzcd}
G(D) \arrow[r, "\eta_{G(D)}"] \arrow[dr, "1_{G(D)}"'] \arrow[ddr, "G(q)"'] & GFG(D) \arrow[d, "G(\varepsilon_D)"] \\
& G(D) \arrow[d, "G(q)"] \\
& G(D')
\end{tikzcd}
\qquad
\begin{tikzcd}
G(D) \arrow[r, "\eta_{G(D)}"] \arrow[d, "G(q)"'] & GFG(D) \arrow[d, "GFG(q)"] \\
G(D') \arrow[r, "\eta_{G(D')}"] \arrow[dr, "1_{G(D')}"'] & GFG(D') \arrow[d, "G(\varepsilon_{D'})"] \\
& G(D')
\end{tikzcd}$$

therefore $q \circ \varepsilon_D = \varepsilon_{D'} \circ FG(q) : \eta_{G(D)} \longrightarrow G(q)$ in $(G(D) \Rightarrow G)$ because $\eta_{G(D)}$ in initial in $(G(D) \Rightarrow G)$.
Triangle identities. Owing to the way ε_D was defined for every $D \in Ob(\mathcal{D})$, one of the two triangle identities (2.4) holds straightforwardly. As for the other, we can consider the commutative diagrams

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[dr, "\eta_C"'] & GF(C) \arrow[d, "G(1_{F(C)})"] \\
& GF(C)
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[d, "\eta_C"'] & GF(C) \arrow[d, "GF(\eta_C)"] \\
GF(C) \arrow[r, "\eta_{GF(C)}"] \arrow[dr, "1_{GF(C)}"'] & GFGF(C) \arrow[d, "G(\varepsilon_{F(C)})"] \\
& GF(C)
\end{tikzcd}$$

hence $\varepsilon_{F(C)} \circ F(\eta_C) = 1_{F(C)} : F(C) \longrightarrow F(C)$, because η_C is initial in $(C \Rightarrow G)$, then also the second triangle identity holds.
Uniqueness. Let's suppose that $\varepsilon, \varepsilon' : F \circ G \longrightarrow 1_{\mathcal{D}}$ are natural transformations and that both (η, ε) and (η, ε') satisfy the triangle identities (2.4). Let $D \in Ob(\mathcal{D})$. It follows that the triangle

$$\begin{tikzcd}
G(D) \arrow[r, "\eta_{G(D)}"] \arrow[dr, "1_{G(D)}"'] & GFG(D) \arrow[d, "G(\varepsilon_D)"] \\
& G(D)
\end{tikzcd}$$

commutes and so ε_D is a map from $(FG(D), \eta_{G(D)})$ to $(D, 1_{G(D)})$ in $(G(D) \Rightarrow G)$. Exactly for the same reason, ε'_D is too. Due to the initiality of $\eta_{G(D)}$, they must be equal, and by the arbitrariness of $D \in Ob(\mathcal{D})$, we infer that $\varepsilon = \varepsilon'$. □

Lemma 2.1. *Let $\mathcal{C}$, $\mathcal{D}$ be two categories and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. If F has a right [respectively left] adjoint, then it is unique.*

Proof. We prove only the uniqueness of the right adjoint to F, since the uniqueness of the left adjoint follows suit. Let's suppose that $G, G' : \mathcal{D} \longrightarrow \mathcal{C}$ are right adjoints to F. Then for every $C \in Ob(\mathcal{C})$ and for every $D \in Ob(\mathcal{D})$ there are natural isomorphisms

$$\mathcal{D}(F(C), D) \overset{\varphi_{C,D}}{\cong} \mathcal{C}(C, G(D)) \text{ and } \mathcal{D}(F(C), D) \overset{\psi_{C,D}}{\cong} \mathcal{C}(C, G'(D))$$

and so $\mathcal{C}(C, G(D)) \overset{\psi_{C,D} \circ \varphi^{-1}_{C,D}}{\cong} \mathcal{C}(C, G'(D))$ is a natural isomorphism too. We need to show that $G(D) \simeq G'(D)$.
If we consider $1_{G(D)} : G(D) \longrightarrow G(D)$ in $\mathcal{D}$, we deduce that

$$1_{G(D)} \cong \varphi^{-1}_{G(D),D}(1_{G(D)}) \cong \psi_{G(D),D}(\varphi^{-1}_{G(D),D}(1_{G(D)})) : G(D) \longrightarrow G'(D),$$

and if we consider $1_{G'(D)} : G'(D) \longrightarrow G'(D)$ in $\mathcal{D}$, we deduce that

$$1_{G'(D)} \cong \psi^{-1}_{G'(D),D}(1_{G'(D)}) \cong \varphi_{G'(D),D}(\psi^{-1}_{G'(D),D}(1_{G'(D)})) : G'(D) \longrightarrow G(D).$$

For the sake of neatness we omit the subscripts, for instance we write φ in place of $\varphi_{G(D),D}$. By naturality the following diagram commutes for every $p : C' \longrightarrow C$ in $\mathcal{C}$:

$$\begin{tikzcd} \mathcal{D}(F(C),D) \arrow[r, "-\circ F(p)"] \arrow[d, "\varphi"'] & \mathcal{D}(F(C'),D) \arrow[d, "\varphi"] \\ \mathcal{C}(C,G(D)) \arrow[r, "-\circ p"'] & \mathcal{C}(C',G(D)) \end{tikzcd}$$

hence for $p \equiv \psi(\varphi^{-1}(1_{G(D)}))$, and by applying the commutative square above to $\psi^{-1}(1_{G'(D)})$, we have:

$$\varphi(\psi^{-1}(1_{G'(D)})) \circ \psi(\varphi^{-1}(1_{G(D)})) = \varphi[\psi^{-1}(1_{G'(D)}) \circ F(\psi(\varphi^{-1}(1_{G(D)})))] \tag{2.5}$$

Similarly, by naturality the following diagram commutes for every $p : C' \longrightarrow C$ in $\mathcal{C}$:

$$\begin{tikzcd} \mathcal{C}(C,G'(D)) \arrow[r, "-\circ p"] \arrow[d, "\psi^{-1}"'] & \mathcal{C}(C',G'(D)) \arrow[d, "\psi^{-1}"] \\ \mathcal{D}(F(C),D) \arrow[r, "-\circ F(p)"'] & \mathcal{D}(F(C'),D) \end{tikzcd}$$

hence for $p \equiv \psi(\varphi^{-1}(1_{G(D)}))$, and by applying the commutative square above to $1_{G'(D)}$, we have:

$$\psi^{-1}(1_{G'(D)}) \circ F(\psi(\varphi^{-1}(1_{G(D)}))) = \psi^{-1}(1_{G'(D)} \circ \psi(\varphi^{-1}(1_{G(D)}))) \tag{2.6}$$

Finally, we gather that $\varphi(\psi^{-1}(1_{G'(D)})) \circ \psi(\varphi^{-1}(1_{G(D)})) \overset{(2.5)}{=} \varphi[\psi^{-1}(1_{G'(D)}) \circ F(\psi(\varphi^{-1}(1_{G(D)})))] \overset{(2.6)}{=} \varphi[\psi^{-1}(1_{G'(D)} \circ \psi(\varphi^{-1}(1_{G(D)})))] = \varphi[\psi^{-1}(\psi(\varphi^{-1}(1_{G(D)})))] = \varphi[\varphi^{-1}(1_{G(D)})] = 1_{G(D)}$. Similarly we can also show that $\psi(\varphi^{-1}(1_{G(D)})) \circ \varphi(\psi^{-1}(1_{G'(D)})) = 1_{G'(D)}$. Concluding: $G(D) \simeq G'(D)$. □

Example 2.1. Forgetful and free functors are usually one the adjoint of the other. For instance, given a field k, there is an adjunction

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{Vect}_k \arrow[l, bend left, "U"] \end{tikzcd}$$

Let S be a set and V a vector space. Let $g : F(S) \longrightarrow V$ be a linear map in $\mathbf{Vect}_k$ and $f : S \longrightarrow U(V)$ be a map in $\mathbf{Set}$. Then we can define the isomorphism

$$\begin{array}{rccc} \bar{\cdot}: & \mathbf{Vect}_k(F(S),V) & \longrightarrow & \mathbf{Set}(S,U(V)) \\ & g & \mapsto & \overline{g} \\ & \overline{f} & \leftarrow & f \end{array}$$

where $\overline{g}(s) \coloneqq g(s)$ for all $s \in S$ and $\overline{f}(\sum_{s\in S} \lambda_s s) \coloneqq \sum_{s\in S} \lambda_s f(s)$ for all formal linear combinations $\sum_{s\in S} \lambda_s s \in F(S)$. These two maps are one the inverse of the other, indeed for any linear map $g : F(S) \longrightarrow V$ in $\mathbf{Vect}_k$:

$$\overline{\overline{g}}(\sum_{s\in S} \lambda_s s) = \sum_{s\in S} \lambda_s \overline{g}(s) = \sum_{s\in S} \lambda_s g(s) = g(\sum_{s\in S} \lambda_s s)$$

for all $\sum_{s\in S} \lambda_s s \in F(S)$, and for any map $f : S \longrightarrow U(V)$:

$$\overline{\overline{f}}(s) = \overline{s} = f(s)$$

for all $s \in S$. Furthermore, the unit $\eta : 1_{\mathbf{Set}} \longrightarrow U \circ F$ is defined as

$$\begin{array}{rccl} \eta_S : & S & \longrightarrow & UF(S) = \{\text{formal linear sums } \sum_{s\in S} \lambda_s s\} \\ & s & \longmapsto & s \end{array}$$

and the counit $\varepsilon : F \circ U \longrightarrow 1_{\mathbf{Vect}_k}$ is defined as

$$\begin{array}{rccc} \varepsilon_V : & FU(V) = \text{vector space of the formal linear sums } \sum_{v\in V} \lambda_v v & \longrightarrow & V \\ & \sum_{v\in V} \lambda_v v & \longmapsto & \text{actual value of } \sum_{v\in V} \lambda_v v \text{ in } V \end{array}$$

Theorem (2.1) claims in particular that an adjunction between two categories $\mathcal{C}$ and $\mathcal{D}$ is a quadruple $(F, G, \eta, \varepsilon)$ where $F : \mathcal{C} \longrightarrow \mathcal{G}$, $G : \mathcal{D} \longrightarrow \mathcal{C}$ are functors and $\eta : 1_{\mathcal{C}} \longrightarrow G \circ F$, $\varepsilon : F \circ G \longrightarrow 1_{\mathcal{D}}$ are natural transformations satisfying the triangle identities. This definition resembles really closely the definition of equivalence. However they are not interchangeable!

Lemma 2.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $(F, G, \eta, \varepsilon)$ be an adjunction between them. Let $\mathbf{Fix}(GF)$ be the full subcategory of $\mathcal{C}$ whose objects are those $C \in Ob(\mathcal{C})$ such that η_C is an isomorphism and let $\mathbf{Fix}(FG)$ be the full subcategory of $\mathcal{D}$ whose objects are those $D \in Ob(\mathcal{D})$ such that ε_D is an isomorphism. Then the adjunction $(F, G, \eta, \varepsilon)$ restricts to an equivalence $(F', G', \eta', \varepsilon')$ between $\mathbf{Fix}(GF)$ and $\mathbf{Fix}(FG)$.*

Proof. By definition of adjunction $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{D} \longrightarrow \mathcal{C}$ are functors and $\eta : 1_{\mathcal{C}} \longrightarrow G \circ F$, $\varepsilon : F \circ G \longrightarrow 1_{\mathcal{D}}$ are natural transformations satisfying the triangle identities (2.4). We need to prove that $(F', G', \eta', \varepsilon')$ is an equivalence between $\mathbf{Fix}(GF)$ and $\mathbf{Fix}(FG)$. Therefore we need to prove that $F' : \mathbf{Fix}(GF) \longrightarrow \mathbf{Fix}(FG)$, $G' : \mathbf{Fix}(FG) \longrightarrow \mathbf{Fix}(GF)$ are functors and $\eta' : 1_{\mathbf{Fix}(GF)} \longrightarrow G' \circ F'$, $\varepsilon' : F' \circ G' \longrightarrow 1_{\mathbf{Fix}(FG)}$ are natural isomorphisms.

F', G' are functors. It suffices to show that F sends objects of $\mathbf{Fix}(GF)$ into objects of $\mathbf{Fix}(FG)$, because from the fact that both $\mathbf{Fix}(GF)$ and $\mathbf{Fix}(FG)$ are full subcategories we infer that every morphism in the former category is sent by F (and then by F' as well) into a morphism of the latter. Also, F' respects composition and identity, since F does. Let $C \in Ob(\mathbf{Fix}(GF)$, so $\eta_C : C \longrightarrow GF(C)$ is an isomorphism, that is there exists $i : GF(C) \longrightarrow C$ such that $i \circ \eta_C = 1_C$ and $\eta_C \circ i = 1_{GF(C)}$. We notice that also $F(\eta_C)$ is an isomorphism, since by the fact that F respects composition and identity: $F(i) \circ F(\eta_C) = F(i \circ \eta_C) = F(1_C) = 1_{F(C)}$ and similarly $F(\eta_C) \circ F(i) = 1_{F(GF(C))}$. By one of the triangle identities (2.4): $\varepsilon_{F(C)} \circ F(\eta_C) = 1_{F(C)}$. Also, by composing to the left with $F(\eta_C)$ and to the right with $F(i)$, we get $F(\eta_C) \circ \varepsilon_{F(C)} = 1_{FGF(C)}$. Hence, also $\epsilon_{F(C)}$ is an isomorphism. By definition: $F(C) \in Ob(\mathbf{Fix}(FG))$, hence we conclude that F restricts to a functor from $\mathbf{Fix}(GF)$ to $\mathbf{Fix}(FG)$.

η' and ε' are natural isomorphisms. We need to prove that η and ε restrict to natural isomorphisms. However this follow straightforwardly from the fact that $\mathbf{Fix}(GF)$ and $\mathbf{Fix}(FG)$ are subcategories. □

Lemma 2.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $(F, G, \eta, \varepsilon)$ be an equivalence between them. Then F is left adjoint to G ($F \dashv G$).*

Proof. Let's consider $(F, G, \eta, \varepsilon)$ an equivalence of categories, so $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ are functors, $\eta : 1_{\mathcal{C}} \longrightarrow G \circ F$ and $\varepsilon : F \circ G \longrightarrow 1_{\mathcal{D}}$ are natural isomorphisms, i.e. $\eta_C : C \longrightarrow GF(C)$ and $\varepsilon_D : FG(D) \longrightarrow D$ are isomorphisms in $\mathcal{C}$ and $\mathcal{D}$ respectively for every $C \in Ob(\mathcal{C})$ and $D \in Ob(\mathcal{D})$.

We need to show that there is a quadruplet $(F, G, \eta', \varepsilon')$ such that $\eta' : 1_{\mathcal{C}} \longrightarrow G \circ F$ and $\varepsilon' : F \circ G \longrightarrow 1_{\mathcal{D}}$ are natural transformations satisfying the triangle identities (2.4). It suffices to define $\eta'_C \equiv \eta_C$ for every $C \in Ob(\mathcal{C})$ and ε'_D given by the composition

$$FG(D) \overset{FG(\varepsilon_D^{-1})}{\longrightarrow} FG(FG(D)) \overset{F(\eta_{G(D)}^{-1})}{\longrightarrow} FG(D) \overset{\varepsilon_D}{\longrightarrow} D$$

for every $D \in Ob(\mathcal{C})$. Then it is straightforward to see that η' and ε' are natural transformations and that they respect the triangle identities (2.4). □

Chapter 3

Limits and colimits

The notions of limits and colimits are one the dual of the other. They provide a way of describing what is going on inside categories and formalizing many common mathematical constructions.

3.1 Limits

Definition 3.1. *Let $\mathcal{C}$ be a category and $\mathbf{I}$ a small category. A functor $D : \mathbf{I} \longrightarrow \mathcal{C}$ is called a **diagram** in $\mathcal{C}$ **of shape** $\mathbf{I}$.*

Definition 3.2. *Let $\mathcal{C}$ be a category, $\mathbf{I}$ a small category and $D : \mathbf{I} \longrightarrow \mathcal{C}$ a diagram in $\mathcal{C}$. A **cone** on D is a pair*

$$(C, (f_I : C \longrightarrow D(I))_{I \in Ob(\mathbf{I})})$$

*where C is an object in $\mathcal{C}$, called the **vertex** of the cone, and $(f_I)_{I \in Ob(\mathbf{I})}$ is a family of maps in $\mathcal{C}$ such that the triangle*

$$\begin{tikzcd} & D(I) \arrow[d, "Du"] \\ C \arrow[ur, "f_I"] \arrow[dr, "f_J"'] & D(J) \end{tikzcd}$$

for all maps $u : I \longrightarrow J$ in $\mathbf{I}$.

In fact, we simply say that a cone on a diagram $D : \mathbf{I} \longrightarrow \mathcal{C}$ is given by a family of maps $(f_I : C \longrightarrow D(I))_{I \in Ob(\mathbf{I})}$ satisfying the property above.

Definition 3.3. *Let $\mathcal{C}$ be a category, $\mathbf{I}$ a small category and $D : \mathbf{I} \longrightarrow \mathcal{C}$ a diagram in $\mathcal{C}$. A **limit** of D is a cone $(p_I : L \longrightarrow D(I))_{I \in Ob(\mathbf{I})}$ with the property that for any cone*

$$(f_I : C \longrightarrow D(I))_{I \in Ob(\mathbf{I})}$$

*there exists a unique map $\overline{f} : C \longrightarrow L$ in $\mathcal{C}$ such that $p_I \circ \overline{f} = f_I$ for all $I \in Ob(\mathbf{I})$. The maps p_I are called the **projections** of the limit.*

It is very common to abuse language by referring to L as the limit of D and by indicating it by $L = \lim_{\leftarrow \mathbf{I}} D$. Emphatically, the cone $(p_I : L \longrightarrow D(I))_{I \in Ob(\mathbf{I})}$ is called the limit cone.

We want to give some examples that limits in categories allow us to construct.

We recall that a category is discrete if it has only the identities as maps. Let's consider the case in which $\mathbf{I}$ is the discrete category on a set I, that is a category whose objects are the elements of the set I and whose maps are the identities only. A diagram D of shape I in a category $\mathcal{C}$ is simply a family $(X_i)_{i \in I}$ of objects in $\mathcal{C}$, and a cone on D is given by an object $C \in Ob(\mathcal{C})$ together with maps $(f_i : C \longrightarrow X_i)$ for each $i \in I$. A limit of D turns out to be a product.

Definition 3.4. *Let $\mathcal{C}$ be a category, I a set and $(X_i)_{i\in I}$ a family of objects of $\mathcal{C}$. A **product** of $(X_i)_{i\in I}$ consists of an object P and a family of maps $(P \xrightarrow{p_i} X_i)_{i\in I}$, such that for all objects C and families of maps $(C \xrightarrow{f_i} X_i)_{i\in I}$ there exists a unique map $\overline{f}: C \longrightarrow P$ such that $p_i \circ \overline{f} = f_i$ for all $i \in I$.*

When the product exists, it is common to denote P as $\prod_{i\in I} X_i$. In the particular case in which we consider $\mathbf{I}$ to be the discrete category $\mathbf{2}$ on the set of two elements, we end up finding the binary product.

Definition 3.5. *Let $\mathcal{C}$ be a category and $X, Y \in Ob(\mathcal{C})$. The **binary product** of X and Y consists of an object P and maps*

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

such that for all objects and maps

$$\begin{tikzcd} & C \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X & & Y \end{tikzcd}$$

in the category $\mathcal{C}$, there exists a unique map $\overline{f}: C \longrightarrow P$ such that

$$\begin{tikzcd} & C \arrow[d, dashed, "\overline{f}"] \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] & \\ & P \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ X & & Y \end{tikzcd}$$

*commutes. The maps p_1 and p_2 are called the **projections**.*

Let's consider the case in which $\mathbf{I}$ is the category $\mathbf{E}$ with two objects and two morphisms from the first to the second object, aside from the identities. A diagram D of shape $\mathbf{E}$ in a category $\mathcal{C}$ is a pair $X \overset{s}{\underset{t}{\rightrightarrows}} Y$ of maps in $\mathcal{C}$, and a cone on D is given by objects and maps

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, shift left, "s"] \arrow[rr, shift right, "t"'] & & Y \end{tikzcd}$$

such that $s \circ f = g$ and $t \circ f = g$. However, it is clear that g is completely determined by f, so it needs not be introduced.

Definition 3.6. *Let $\mathcal{C}$ be a category. A **fork** in $\mathcal{C}$ consists of objects and maps $C \xrightarrow{f} X \overset{s}{\underset{t}{\rightrightarrows}} Y$ such that $s \circ f = t \circ f$.*

It follows that a cone on a diagram D of shape $\mathbf{E}$ turns out to simply be an object $C \in Ob(\mathcal{C})$ and a map $f: C \longrightarrow X$ in $\mathcal{C}$ such that

$$C \xrightarrow{f} X \overset{s}{\underset{t}{\rightrightarrows}} Y$$

is a fork. A limit of D is in this case a fork with some universal property, often regarded as an equalizer.

Definition 3.7. *Let $\mathcal{C}$ be a category and* $X \overset{s}{\underset{t}{\rightrightarrows}} Y$ *objects and maps in $\mathcal{C}$. An **equalizer** of s and t is an object E together with a map $i : E \longrightarrow X$ in $\mathcal{C}$ such that*

$$E \xrightarrow{i} X \overset{s}{\underset{t}{\rightrightarrows}} Y$$

is a fork and such that for any other fork $C \xrightarrow{f} X \overset{s}{\underset{t}{\rightrightarrows}} Y$ *there exists a unique map $\overline{f} : C \longrightarrow E$ such that*

$$\begin{tikzcd} C \arrow[d, dashed, "\overline{f}"'] \arrow[dr, "f"] & \\ E \arrow[r, "i"'] & X \end{tikzcd}$$

commutes.

As third and last example, let's consider the case in which $\mathbf{I}$ is the category $\mathbf{P}$, where there are three objects and a morphism from each of first two to the third one, aside from the identities. A diagram D of shape $\mathbf{P}$ in a category $\mathcal{C}$ consists of objects and maps

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

in $\mathcal{C}$, and a cone on D is given by objects and maps

$$\begin{tikzcd} & X \arrow[d, "s"] \\ C \arrow[ur, "f_X"] \arrow[dr, "f_Z"'] & Z \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[d, "t"] \\ C \arrow[ur, "f_Y"] \arrow[dr, "f_Z"'] & Z \end{tikzcd}$$

such that $s \circ f_X = f_Z$ and $t \circ f_Y = f_Z$. However, it is clear that f_Z is completely determined by f_X and f_Y, so it needs not be introduced. It follows that a cone on a diagram D of shape $\mathbf{P}$ turns out to simply be an object $C \in Ob(\mathcal{C})$ and maps $f_1 : C \longrightarrow X$, $f_2 : C \longrightarrow Y$ in $\mathcal{C}$ such that

$$\begin{tikzcd} C \arrow[r, "f_2"] \arrow[d, "f_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

is a commutative square, i.e. $s \circ f_1 = t \circ f_2$. A limit of D is in this case a commutative diagram with some universal property, often regarded as a pullback.

Definition 3.8. *Let $\mathcal{C}$ be a category and*

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

*objects and maps in $\mathcal{C}$. A **pullback** of this diagram consists of an object $P \in Ob(\mathcal{C})$ and maps $p_1 : P \longrightarrow X$, $p_2 : P \longrightarrow Y$ such that*

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

commutes, and such that for any commutative square

$$\begin{tikzcd} C \arrow[r, "f_2"] \arrow[d, "f_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

in $\mathcal{C}$, there is a unique map $\overline{f} : C \longrightarrow P$ in $\mathcal{C}$ such that

$$\begin{tikzcd} C \arrow[dr, "\overline{f}"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ & X \arrow[r, "s"'] & Z \end{tikzcd}$$

commutes, which means $p_1 \circ \overline{f} = f_1$ and $p_2 \circ \overline{f} = f_2$.

3.2 Colimits

The dual of the notion of limit is the notion of colimit.

Definition 3.9. *Let $\mathcal{C}$ be a category, $\mathbf{I}$ a small category and $D : \mathbf{I} \longrightarrow \mathcal{C}$ a diagram in $\mathcal{C}$. Let $D^{op} : \mathbf{I}^{op} \longrightarrow \mathcal{C}^{op}$ be the dual functor of the diagram D. A* ***cocone*** *on D is a cone on D^{op}, and a* ***colimit*** *of D is a limit of D^{op}.*

If we want to state it explicitly, a cocone on a diagram $D : \mathbf{I} \longrightarrow \mathcal{C}$ in $\mathcal{C}$ is given by an object $C \in Ob(\mathcal{C})$, called the vertex of the cocone, and by a family

$$(f_I : D(I) \longrightarrow C)_{I \in \mathbf{I}}$$

of maps in $\mathcal{C}$ such that for all maps $u : I \longrightarrow J$ in $\mathbf{I}$ the diagram

$$\begin{tikzcd} D(I) \arrow[dd, "Du"'] \arrow[dr, "f_I"] & \\ & C \\ D(J) \arrow[ur, "f_J"'] & \end{tikzcd}$$

commutes. A colimit of D is a cocone

$$(p_I : D(I) \longrightarrow L)_{I \in \mathbf{I}}$$

such that for any cocone $(f_I : D(I) \longrightarrow C)_{I \in \mathbf{I}}$ there exists a unique map $\overline{f} : L \longrightarrow C$ such that $\overline{f} \circ p_I = f_I$ for all $I \in \mathbf{I}$.

Now, we want to see how the previous examples dualize, giving rise to some examples of colimits.

The dual of a product is called a coproduct or a sum, hence it is a colimit of shape $\mathbf{I}$ for some discrete category $\mathbf{I}$. As in the case of products, we can give the following definition.

Definition 3.10. *Let $\mathcal{C}$ be a category, I a set and $(X_i)_{i \in I}$ a family of objects of $\mathcal{C}$. A* ***sum*** *of $(X_i)_{i \in I}$ consists of an object S and a family of maps $(X_i \xrightarrow{p_i} S)_{i \in I}$, such that for all objects C and families of maps $(X_i \xrightarrow{f_i} C)_{i \in I}$ there exists a unique map $\overline{f} : S \longrightarrow C$ such that $\overline{f} \circ p_i = f_i$ for all $i \in I$.*

When the sum exists, it is commonly denoted by $\sum_{i\in I} X_i$ or by $\coprod_{i\in I} X_i$. Let's consider the case in which I is the set of two elements.

Definition 3.11. *Let $\mathcal{C}$ be a category and $X, Y \in Ob(\mathcal{C})$. The* ***binary sum*** *of X and Y consists of an object S and maps*

$$\begin{tikzcd} X \arrow[dr, "p_1"] & & Y \arrow[dl, "p_2"'] \\ & S & \end{tikzcd}$$

such that for all objects and maps

$$\begin{tikzcd} X \arrow[dr, "f_1"] & & Y \arrow[dl, "f_2"'] \\ & C & \end{tikzcd}$$

in the category $\mathcal{C}$, there exists a unique map $\overline{f} : S \longrightarrow C$ such that

$$\begin{tikzcd} X \arrow[dr, "f_1"] \arrow[ddr, "f_1"'] & & Y \arrow[dl, "f_2"'] \arrow[ddl, "f_2"] \\ & S \arrow[d, dashed, "\overline{f}"] & \\ & C & \end{tikzcd}$$

commutes. The maps p_1 and p_2 are called the ***coprojections****.*

Dualizing the definition of equalizer, we obtain the definition of coequalizer, which is an instance of a colimit of shape $\mathbf{E}$. We recall that $\mathbf{E}$ is the category with two objects and two morphisms from the first to the second object, aside from the identities.

Definition 3.12. *Let $\mathcal{C}$ be a category and $X \overset{s}{\underset{t}{\rightrightarrows}} Y$ objects and maps in $\mathcal{C}$. A* ***coequalizer*** *of s and t is an object E together with a map $i : Y \longrightarrow E$ in $\mathcal{C}$ such that $i \circ s = i \circ t$ and such that for any other diagram $X \overset{s}{\underset{t}{\rightrightarrows}} Y \xrightarrow{f} C$ for which $f \circ s = f \circ t$ there exists a unique map $\overline{f} : E \longrightarrow C$ such that*

$$\begin{tikzcd} Y \arrow[d, "i"'] \arrow[dr, "f"] & \\ E \arrow[r, dashed, "\overline{f}"'] & C \end{tikzcd}$$

commutes.

Finally, we look into the dual of a pullback. We recall that $\mathbf{P}$ is the category where there are three objects and a morphism from each of first two to the third one, aside from the identities. A colimit of shape $\mathbf{P}^{op}$ is called a pushout.

Definition 3.13. *Let $\mathcal{C}$ be a category and*

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \\ Z & \end{tikzcd}$$

*objects and maps in $\mathcal{C}$. A **pushout** of this diagram consists of an object $P \in Ob(\mathcal{C})$ and maps $p_1 : Y \longrightarrow P$, $p_2 : Z \longrightarrow P$ such that*

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \arrow[d, "p_1"] \\ Z \arrow[r, "p_2"'] & P \end{tikzcd}$$

commutes, and such that for any commutative square

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \arrow[d, "f_1"] \\ Z \arrow[r, "f_2"'] & C \end{tikzcd}$$

in $\mathcal{C}$, there is a unique map $\overline{f} : P \longrightarrow C$ in $\mathcal{C}$ such that

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \arrow[d, "p_1"] \arrow[ddr, bend left, "f_1"] & \\ Z \arrow[r, "p_2"'] \arrow[drr, bend right, "f_2"'] & P \arrow[dr, dashed, "\overline{f}"] & \\ & & C \end{tikzcd}$$

commutes, which means $\overline{f} \circ p_1 = f_1$ and $\overline{f} \circ p_2 = f_2$.

3.3 Monics and epics

Let's consider a function between sets. As it is well-known, studying its being injective and surjective is crucial for many properties of the function. For a map in an arbitrary category, though, these two notions do not make any sense, since they are too specific. It is curious that the generalization of injectivity and surjectivity to the more general context of categories reveals the complexity these notions hide: the generalization of injectivity, as well as that one of surjectivity, gives three different new properties.

Definition 3.14. *Let $\mathcal{C}$ be a category. A map $f : X \longrightarrow Y$ in $\mathcal{C}$ is **monic** (or a **monomorphism**) if for every $A \in Ob(\mathcal{C})$ and every maps* $\begin{tikzcd} A \arrow[r, shift left, "x"] \arrow[r, shift right, "x'"'] & X \end{tikzcd}$ *:*

$$f \circ x = f \circ x' \Rightarrow x = x'$$

The notion of monomorphism is the generalization of the notion of injection. Indeed, a map in **Set** can be proved to be monic if and only if it is injective. Indeed, if f is an injective map, then the condition of the above definition surely holds and so it is monic; conversely, if f is a monomorphism, then it suffices to take $A \equiv 1$ in the definition of monomorphism to see that is is also injective.

Lemma 3.1. *A map $f : X \longrightarrow Y$ is monic if and only if the following square is a pullback:*

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{3.1}$$

Proof. ($\Rightarrow$) Let's suppose that $f : X \longrightarrow Y$ is monic in the category $\mathcal{C}$. Therefore for any object $C \in Ob(\mathcal{C})$ and any maps $C \overset{x}{\underset{x'}{\rightrightarrows}} X$, we have that $f \circ x = f \circ x' \Longrightarrow x = x'$. Let's now show that the diagram (3.1) is a pullback. First of all it commutes, since trivially: $f \circ 1 = f \circ 1$. Let's now consider an arbitrary object $C \in Ob(\mathcal{C})$ and maps $C \overset{x}{\underset{x'}{\rightrightarrows}} X$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, "x'"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Therefore we have that $f \circ x = f \circ x'$. By the fact that f is monic, it follows that $x = x'$. We set $\overline{f} = x$, and by doing so we get a map $\overline{f} : C \longrightarrow X$ such that $1 \circ \overline{f} = 1 \circ x = x$ and $1 \circ \overline{f} = 1 \circ x = x = x'$, and this map is unique since for every possible other map $x'' : C \longrightarrow X$ in $\mathcal{C}$ we can go through the same argument with x'' in place of x' and deduce that $\overline{f} = x = x''$. Therefore the diagram (3.1) is a pullback.

($\Leftarrow$). Let's suppose that the diagram (3.1) is a pullback. Therefore for any object $C \in Ob(\mathcal{C})$ and any maps $C \overset{x}{\underset{x'}{\rightrightarrows}} X$ such that the square

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, "x'"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{3.2}$$

commutes, there exists a unique map $\overline{f} : C \longrightarrow X$ such that $1 \circ \overline{f} = x$ and $1 \circ \overline{f} = x'$. However this means that $\overline{f} = x = x'$. Also, we are assuming that the diagram (3.2) is commutative, so we are assuming that $f \circ x = f \circ x'$. Concluding, taking arbitrary $C \in Ob(\mathcal{C})$ and arbitrary $C \overset{x}{\underset{x'}{\rightrightarrows}} X$ in $\mathcal{C}$ and supposing $f \circ x = f \circ x'$, we have deduced that $x = x'$, thus $f \in \mathcal{C}(X, Y)$ is monic. □

The notion of monomorphism can have different shades, indeed we can define also what follows.

Definition 3.15. *Let $\mathcal{C}$ be a category. A map $f : X \longrightarrow Y$ in $\mathcal{C}$ is* ***regular monic*** *(or a* ***regular monomorphism****) if there is an object $Z \in Ob(\mathcal{C})$ and maps $Y \rightrightarrows Z$ of which f is an equalizer.*

Definition 3.16. *Let $\mathcal{C}$ be a category. A map $f : X \longrightarrow Y$ in $\mathcal{C}$ is* ***split monic*** *(or a* ***split monomorphism****) if there exists a map $e : Y \longrightarrow X$ such that $e \circ f = 1_X$.*

Proposition 3.1. *In any category $\mathcal{C}$ the following implications hold:*

$$\textit{split monic} \;\Rightarrow\; \textit{regular monic} \;\Rightarrow\; \textit{monic}$$

Proof. Let's consider the map $m : A \longrightarrow B$ in a category $\mathcal{C}$ and let's suppose that it is split monic. Therefore there exists a map $e : B \longrightarrow A$ in $\mathcal{C}$ such that

$$e \circ m = 1_A \tag{3.3}$$

Then m is also regular monic, since it is the equalizer of $B \overset{m \circ e}{\underset{1_B}{\rightrightarrows}} B$ (we are taking $C \equiv B$ in the definition of regular monomorphism). Indeed, $A \xrightarrow{m} B \overset{m \circ e}{\underset{1_B}{\rightrightarrows}} B$ is a fork: $(m \circ e) \circ m =$

$m \circ (e \circ m) \overset{(3.3)}{=} m \circ 1_A = m = 1_B \circ m$. Let $X \in Ob(\mathcal{C})$ be arbitrary and let's consider a fork $X \xrightarrow{f} B \overset{m \circ e}{\underset{1_B}{\rightrightarrows}} B$ with $f \in \mathcal{C}(X, B)$. Being it a fork, we have:

$$m \circ e \circ f = 1_B \circ f \tag{3.4}$$

Then the map $\overline{f} := (e \circ f) \in \mathcal{C}(X, A)$ is such that $m \circ \overline{f} = f$ since $m \circ \overline{f} = m \circ e \circ f \overset{(3.4)}{=} 1_B \circ f = f$, and $\overline{f}$ is uniquely determined by e and f due to the way it is defined (one can argue that e is not unique; however in order to maintain the validity of the previous equalities, we can in general only take the exact e which appears in the fork $A \xrightarrow{m} B \overset{m \circ e}{\underset{1_B}{\rightrightarrows}} B$). Therefore m is in fact also a regular monomorphism.
Let's consider now the map $m : A \longrightarrow B$ in a category $\mathcal{C}$ and let's suppose that it is regular monic. Therefore there exist an object $C \in Ob(\mathcal{C})$ and maps $B \rightrightarrows C$ in $\mathcal{C}$ of which m is an equalizer. This means that $A \xrightarrow{m} B \overset{y}{\underset{y'}{\rightrightarrows}} C$ is a fork, i.e.:

$$y \circ m = y' \circ m \tag{3.5}$$

and the universal property of the equalizers hold. Then m is also a monomorphism in $\mathcal{C}$. In order to show this, let's consider an arbitrary object $X \in Ob(\mathcal{C})$ and maps $X \overset{x}{\underset{x'}{\rightrightarrows}} A$ such that

$$m \circ x = m \circ x' \tag{3.6}$$

We can consider the diagram $X \xrightarrow{m \circ x} B \overset{y}{\underset{y'}{\rightrightarrows}} C$. This is a fork since $y \circ m = y' \circ m$ by (3.5) and thus $y \circ (m \circ x) = y \circ m \circ x = y' \circ m \circ x = y' \circ (m \circ x)$. Therefore, by the universal property of the equalizers, there must exist a unique map $\overline{f} : X \longrightarrow A$ in $\mathcal{C}$ such that $m \circ \overline{f} = m \circ x \overset{(3.6)}{=} m \circ x'$, and by the uniqueness of $\overline{f}$ it must be that $\overline{f} = x = x'$. Therefore $x = x'$ and m is effectively monic. □

Example 3.1. In **Ab**, it is possible to show that all monics are regular but not all monics are split.

We can dualize the notions of monic, regular monic and spit monic as follows.

Definition 3.17. *Let $\mathcal{C}$ be a category. A map $f : X \longrightarrow Y$ is **epic** (o a **epimorphism**) if for all objects $Z \in Ob(\mathcal{C})$ and maps $Y \overset{x}{\underset{x'}{\rightrightarrows}} Z$:*

$$x \circ f = x' \circ f \Rightarrow x = x'$$

The notion of epimorphism is the generalization of the notion of surjection. Indeed, it is easy to see that in the category **Set** of sets a map is surjetive if and only if it is epic: trivially, surjective implies epic, and to see the converse it suffices to take Z to be the two-sided element {true, false}, x to be the characteristic function of the image of f and x' to be the function with constant value true, in the definition above.

Generally speaking, being surjective and being epic do not coincide: in the category **Ring** of rings for instance, it is possible to see that the inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$ is epic but not surjective. Also, this is an example of a map which is monic and epic, but not an isomorphism!

Definition 3.18. *Let $\mathcal{C}$ be a category. A map $f : X \longrightarrow Y$ in $\mathcal{C}$ is **regular epic** (or a **regular epimorphism**) if there is an object $Z \in Ob(\mathcal{C})$ and maps $Z \rightrightarrows X$ of which f is a coequalizer.*

Definition 3.19. *Let $\mathcal{C}$ be a category. A map $f : X \longrightarrow Y$ in $\mathcal{C}$ is **split epic** (or a **split epimorphism**) if there exists a map $e : Y \longrightarrow X$ such that $f \circ e = 1_Y$.*

Proposition 3.2. *In any category $\mathcal{C}$ the following implications hold:*

$$\textit{epic} \;\Rightarrow\; \textit{regular epic} \;\Rightarrow\; \textit{split epic}$$

Proof. It follows by duality from Proposition (3.1). □

Theorem 3.1. *In any category, a map is an isomorphism if and only if it is both monic and regular epic.*

Proof. Let's consider a map $f : A \longrightarrow B$ in a category $\mathcal{C}$ and let's prove that f is an isomorphism if and only if it is monic and regular epic.
($\Rightarrow$) Let's suppose that f is an isomoprhism in $\mathcal{C}$, therefore there exists $g : B \longrightarrow A$ in $\mathcal{C}$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. Let's take an object $X \in Ob(\mathcal{C})$ and maps $X \underset{x'}{\overset{x}{\rightrightarrows}} A$ in $\mathcal{C}$ such that $f \circ x = f \circ x'$. Then by composing with g it follows that $x = 1_A \circ x = (g \circ f) \circ x = g \circ (f \circ x) = g \circ (f \circ x') = (g \circ f) \circ x' = 1_A \circ x' = x'$, therefore f is monic. Let's consider now the object $A \in Ob(\mathcal{C})$ and maps $A \underset{g\circ f}{\overset{1_A}{\rightrightarrows}} A$ in $\mathcal{A}$. Then f is coequalizer of 1_A and $g \circ f$. First of all, $f \circ 1_A = f \circ (g \circ f)$, therefore $A \underset{g\circ f}{\overset{1_A}{\rightrightarrows}} A \xrightarrow{f} B$ is a fork. In addition, for any object $X \in Ob(\mathcal{C})$ and for any fork $A \underset{g\circ f}{\overset{1_A}{\rightrightarrows}} A \xrightarrow{\xi} X$, there is a unique map $\overline{\xi} := \xi \circ g : B \longrightarrow X$ in $\mathcal{C}$ such that $\overline{\xi} \circ f = \xi$, indeed: $\overline{\xi} \circ f = (\xi \circ g) \circ f = \xi \circ (g \circ f) = \xi \circ 1_A = \xi$, and $\overline{\xi}$ is uniquely determined by g and ξ.
($\Leftarrow$) Let's suppose that f is a monic and regular epic morphism in the category $\mathcal{C}$. We need to find a map $g \in \mathcal{C}(B, A)$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Since f is regular epic, there exist an object $C \in Ob(\mathcal{C})$ and maps $C \underset{x'}{\overset{x}{\rightrightarrows}} A$ for which f is a coequalizer. In particular, $f \circ x = f \circ x'$, and by the fact that f is monic it follows that $x = x'$. We can consider the diagram $C \underset{x'}{\overset{x}{\rightrightarrows}} A \xrightarrow{1_A} A$, and by what we have just said, it is a fork since $1_A \circ x = x = x' = 1_A \circ x'$. Therefore, the universal property of the coequalizers assures the existence of a unique map $g : B \longrightarrow A$ in $\mathcal{C}$ such that $g \circ f = 1_A$. However it is also true that $f \circ g = 1_B$, indeed from the equalities $f = f \circ 1_A = f \circ (g \circ f) = (f \circ g) \circ f$ we infer that necessarily $f \circ g = 1_B$. Therefore f is an isomoprhism in $\mathcal{C}$. □

It is licit to search for the real reason why the notions of monic, regular monic and split monic, and by duality those ones of epic, regular epic and split epic, collapse into the single notions of injectivity and surjectivity respectively in the category **Set** of sets. Interestingly enough, the answer deals with the axiom of choice, as the following result shows. Of course, we should previously generalize the axiom of choice for any category $\mathcal{C}$.

Definition 3.20. *Let $\mathcal{C}$ be a category. The category $\mathcal{C}$ satisfies the* ***axiom of choice*** *if every epimorphism in $\mathcal{C}$ is a split epic.*

As regards **Set** in particular, the axiom of choice states then that every surjection splits.

Proposition 3.3. *Assuming that the category* **Set** *satisfies the axiom of choice, the following implications hold:*

$$\textit{monic} \;\Leftrightarrow\; \textit{regular monic} \;\Leftrightarrow\; \textit{split monic}$$

and by duality

$$\textit{epic} \;\Leftrightarrow\; \textit{regular epic} \;\Leftrightarrow\; \textit{split epic}$$

Proof. We need only to prove that in **Set** with the axiom of choice every epimorphism is regular and that every regular epimorphism is split.
Let's consider an epimorphism $f : X \longrightarrow Y$ in **Set**. Therefore for all sets Z and functions $Y \underset{y'}{\overset{y}{\rightrightarrows}} Z$ if $y \circ f = y' \circ f$ then $y = y'$. Also, by the fact that in **Set** epics coincide with surjections, we have that f is a section, since the axiom of choice holds: there exists a function $i : Y \longrightarrow X$ such that $f \circ i = 1_Y$. We now claim that f is also a regular epimorphism, which means that there exist a set C and functions $C \underset{c'}{\overset{c}{\rightrightarrows}} X$ for which f is a coequalizer. We claim that it suffices to take $C \equiv X$, $c \equiv i \circ f$ and $c' \equiv 1_X$ and we show then that f is a coequalizer for $X \underset{1_X}{\overset{i \circ f}{\rightrightarrows}} X$. First of all, $X \underset{1_X}{\overset{i \circ f}{\rightrightarrows}} X \xrightarrow{f} Y$ is a fork, since: $f \circ (i \circ f) = (f \circ i) \circ f = 1_Y \circ f$.
Let's consider another fork $X \underset{1_X}{\overset{i \circ f}{\rightrightarrows}} X \xrightarrow{\xi} Z$. Then there exists a unique function $\overline{\xi} : Y \longrightarrow Z$ such that $\overline{\xi} \circ f = \xi$, for it suffices to take $\overline{\xi} = \xi \circ i$: $\overline{\xi} \circ f = (\xi \circ i) \circ f = \xi \circ i \circ f \overset{*}{=} 1_X \circ \xi = \xi$ where $(*)$ is given by the fact that $X \underset{1_X}{\overset{i \circ f}{\rightrightarrows}} X \xrightarrow{\xi} Z$ is a fork, and trivially $\overline{\xi}$ is uniquely determined by ξ and i. Concluding, f is in fact a regular epimorphism.
Let's now suppose that $f : X \longrightarrow Y$ in **Set** is a regular epimorphism, and let's show that it is split epic as well. Therefore we are supposed to find a right inverse of f, i.e. a function $i : Y \longrightarrow X$ in **Set** such that $f \circ i = 1_Y$. However we know that a regular epimorphism is epic, which in the category of **Set** is equivalent to be a surjection. Therefore f is a surjection, and by the axiom of choice it has a section, which is exactly the function i that we were searching for. $\square$

Chapter 4

The categories of Boolean and Heyting algebras

In this chapter Heyting and Boolean algebras are introduced. They are algebraic structures where intuitionistic and classical logic respectively can be interpreted. Moreover, they arise naturally from any given logical language, as the formalization of Lindenbaum algebras will show. We will also try to deal with algebras from a purely categorical point of view: this is the turning point at which categorical logic arises, that is when logic in specific and not only mathematics in general is tackled with a merely categorical mindset.

4.1 Lattices

Definition 4.1. *A **semilattice** is a couple $(X, *)$ where X is a set and $* : X \times X \longrightarrow X$, $(x, y) \longmapsto x * y$ is an operation such that:*

- $x * x = x$;
- $x * y = y * x$;
- $(x * y) * z = x * (y * z)$.

Definition 4.2. *Let $(S, \leq)$ be a poset and $x, y \in S$. The **infimum** of x and y is an element $x \wedge y \in S$ such that:*

- $x \wedge y \leq x$ *and* $x \wedge y \leq y$;
- *for all* $z \in S$ *such that* $z \leq x$ *and* $z \leq y$, *then* $z \leq x \wedge y$.

*Dually, the **supremum** of x and y is an element $x \vee y \in S$ such that:*

- $x \leq x \vee y$ *and* $y \leq x \vee y$;
- *for all* $z \in S$ *such that* $x \leq z$ *and* $y \leq z$, *then* $x \vee y \leq z$.

It is straightforward to show that the supremum and the infimum, when they exist, are unique.

Proposition 4.1. *Let $(X, *)$ be a semilattice. Define $x \leq y \overset{def}{\Leftrightarrow} x * y = x$, then $(X, \leq)$ is a poset in which every pair of elements has an infimum. Conversely, given a poset $(S, \leq)$ where every pair of elements has an infimum, define $x * y$ to be the infimum of x and y, then $(S, *)$ is a semilattice.*

Proof. Let's consider $(X, \leq)$ defined as in the statement of the Proposition; $\leq$ is a partial order:

1. $x * x = x$, thus $x \leq x$;

2. if $x \leq y$ and $y \leq x$ then $x = x * y = y * x = y$;

3. if $x \leq y \leq z$ then $x * z = (x * y) * z = x * (y * z) = x * y = x$, then $x \leq z$.

Let $x, y \in X$. We notice that: $(x * y) * x = x * (y * x) = x * (x * y) = (x * x) * y = x * y$, so $x * y \leq x$. Similarly $x * y \leq y$. Also, for every $z \in X$ such that $z \leq x$ and $z \leq y$ we have that: $z * (x * y) = (z * x) * y = z * y = z$, hence $z \leq x * y$. Therefore $x * y$ is an infimum for $x, y \in X$.

Conversely, let's consider $(S, *)$ as in the second part of the theorem. Then:

- $x * x = x$ since the infimum of an element is the element itself;
- $x * y = y * x$ since the infimum of $\{x, y\}$ is the same of the infimum of $\{y, x\}$;
- let $inf\{-,-\}$ denote the infimum of two elements in S, then $inf\{inf\{x, y\}, z\} = inf\{inf\{x, y\}, inf\{z\}\} = inf\{x, y, z\} = inf\{inf\{x\}, inf\{y, z\}\} = inf\{x, inf\{y, z\}\}$.

Therefore $(S, *)$ is a semilattice. □

A semilattice with the ordering defined in the previos theorem in usually called a **meet-semilattice**, and $\wedge$ substitutes for $*$. If we consider the dual order, setting $y \leq x \overset{def}{\Leftrightarrow} x * y = x$, we usually refer to the associated semilattice as a **join-semilattice**, and $\vee$ substitutes for $*$.

Definition 4.3. *A* ***lattice*** *is a triple $(X, \wedge, \vee)$ where $(X, \wedge)$ is a meet-semilattice and $(X, \vee)$ is a join-semilattice such that they induce the same order on X. More precisely, a lattice is a triple $(X, \wedge, \vee)$ such that:*

(i) $x \wedge x = x$ *and* $x \vee x = x$ *(idempotence);*

(ii) $x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$ *(commutativity);*

(iii) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ *and* $(x \vee y) \vee z = x \vee (y \vee z)$ *(associativity);*

(iv) $x \wedge (x \vee y) = x$ *and* $x \vee (x \wedge y) = x$ *(absorption laws).*

Proposition 4.2. *Let $(X, \wedge, \vee)$ be a lattice. Define $x \leq y \overset{def}{\Leftrightarrow} x \wedge y = x$, then $(X, \leq)$ is a poset in which every pair of elements has an infimum and a supremum. Conversely, given a poset $(S, \leq)$ where every pair of elements has an infimum and a supremum, define $x \wedge y$ to be the infimum of x and y and $x \vee y$ to be the supremum of x and y, then $(S, \wedge, \vee)$ is a lattice.*

Proof. It follows directly from Proposition (4.1), once we have shown that for every $x, y \in X$ $x \wedge y = x \Leftrightarrow x \vee y = y$. And this is true: if $x \wedge y = x$ then $x \vee y = (x \wedge y) \vee y = y$ by the absorption laws, similarly in the other direction. □

Thanks to the previous Proposition, it is possible to think of a lattice as an ordered set where every couple of elements has an infimum and a supremum. For this reason, we interpret the symbols $\wedge$ and $\vee$ as the infimum and the supremum of the elements which they are applied to.

Definition 4.4. *Let $(X, \wedge, \vee)$ be a lattice. A* ***sublattice*** *$(X' \wedge, \vee)$ is a subset $X' \subseteq X$ such that $x \wedge y \in X'$ and $x \vee y \in X'$ for every $x, y \in X'$.*

Definition 4.5. *Let $(X, \wedge, \vee)$ and $(Y, \wedge, \vee)$ be lattices. A map $X \longrightarrow Y$ is said to be a* ***lattice morphism*** *if $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$ for every $x, y \in X$.*

Definition 4.6. *Let $(X, \wedge, \vee)$ be a lattice. It is said to be a* ***distributive lattice*** *if the following two properties hold:*

(i) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee y)$;

(ii) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge y)$.

Definition 4.7. *A lattice $(X, \wedge, \vee)$ is said to be a* ***complete lattice*** *if every subset $Z \subseteq X$ has an infimum and a supremum.*

Every finite subset of a lattice has an infimum and a supremum: by induction it is indeed easy to show that for every $n \in \mathbb{N}$ the subset $\{x_1, ...x_n\}$ of n elements of a lattice has an infimum and a supremum, which we denote respectively $\bigwedge_{i=1}^n x_i$ and $\bigvee_{i=1}^n x_i$. However, there is a priori no guarantee that any infinite subset of a lattice has an infimum or a supremum, but it suffices to assure that it has only one of them to have the other too, as the following result shows.

Proposition 4.3. *Let $(X, \leq)$ be a partially order set in which every subset has an infimum [respectively a supremum]. Then $(X, \wedge, \vee)$ is a complete lattice.*

Proof. It suffices to show that every subset of X has a supremum. Let's consider $Z \subseteq X$ and take $Z' := \{\xi \in X | \forall z \in Z(z \leq \xi)\}$. By hypothesis, Z' has an infimum z'. This is the supremum of the subset Z, indeed: $z \leq z'$ for every $z \in Z$ because $z' \in Z'$, for any $\xi \in X$ such that $\forall z \in Z(z \leq \xi)$ we have that $z' \leq \xi$, because z' is the infimum of Z'. □

We can consider the whole lattice as a subset of itself and in the case of a complete lattice the existence of an infimum and a supremum of the whole lattice is guaranteed. More generally we define:

Definition 4.8. *Let $(X \wedge, \vee)$ be a lattice. An element $z \in X$ is said to be the* ***least element*** *of the lattice if $z \leq x$ for every $x \in X$. Dually, an element $z \in X$ is said to be the* ***greatest element*** *of the lattice if $x \leq z$ for every $x \in X$.*

Of course, if the least and the gretest element in a lattice exist, they are unique. The least and the greatest elements are usually denoted by 0 and 1 respectively.

Definition 4.9. *Let $(X, \wedge, \vee)$ be a lattice with 0 and 1, let $x \in X$. A* ***complement*** *of x is an element $\overline{x} \in X$ such that*

$$x \wedge \overline{x} = 0 \text{ and } x \vee \overline{x} = 1.$$

A lattice with 0 and 1 and with a complement $\overline{x}$ for every $x \in X$ is said to be a ***complemented lattice****.*

The complement of an element needs not be unique, but if the lattice is distributive then the complement is for sure unique.

Proposition 4.4. *Let $(X, \wedge, \vee)$ be a distributive lattice. Every element of X has at most one complement.*

Proof. Let $x \in X$ and let's suppose that $y, y' \in X$ are complements of X. Then $x \wedge y = 0 = x \wedge y'$ and $x \vee y = 1 = x \vee y'$. We have that:

$$y = y \vee 0 = y \vee (x \wedge y') = (y \vee x) \wedge (y \vee y') = 1 \vee (y \vee y') = y \vee y'$$

and similarly $y' = y \vee y'$. Concluding: $y = y'$. □

In a complemented distributive lattice $(X, \wedge, \vee)$, the map $\overline{\cdot} : X \longrightarrow X$, $x \longmapsto \overline{x}$ sending each element in its complement is well-defined, besides it is also self dual since $\overline{\overline{x}} = x$.

Example 4.1. Let X be a set. Its power set $\wp(X)$ is a distributive, complete and complemented lattice with respect to set inclusion $\subseteq$. For every $Z, Z', Z'' \in \wp(X)$:

- $Z \subseteq Z$;
- if $Z \subseteq Z'$ and $Z' \subseteq Z$ then $Z = Z'$;
- if $Z \subseteq Z'$ and $Z' \subseteq Z''$ then $Z \subseteq Z' \subseteq Z''$, and so $Z \subseteq Z''$.

Let $Z, Z' \in \wp(X)$, trivially $Z \subseteq Z \cup Z'$ and $Z' \subseteq Z \cup Z'$. Besides, if $Y \in \wp(X)$ is such that $Z \subseteq Y$ and $Z' \subset Y$ then $Z \cup Z' \subseteq Y$. We can conclude that the join of two elements in $\wp(X)$ exists and is exactly their union $\cup$. Similarly, the meet of two elements in $\wp(X)$ exists and is exatly their interesection $\cap$. More generally, we can consider an arbitrary I-family of subsets $\{Z_i\}_{i\in I}$ of X and we can prove in the same way that $\bigcup_{i\in I} Z_i$ satisfies the conditions of the supremum and $\bigcap_{i\in I} Z_i$ satisfies the conditions of the infimum. As the union and the intersection of an arbitrary family of subset of a subset always exist, $(\wp(X), \subseteq)$ is a complete lattice.
Elementary set theory shows that $\cup$ distributes over $\cap$ and that conversely $\cap$ distributes over $\cup$. Hence, $(\wp(X), \subseteq)$ is distributive.
Trivially, $\varnothing \subseteq Z$ and $Z \subseteq \wp(X)$ for every $Z \in \wp(X)$, thus $\varnothing$ and $\wp(X)$ are the least and greatest elements of the lattice $(\wp(X), \subseteq)$. We denote $\overline{Z} := \wp(X) \setminus Z$ for every $Z \in \wp(X)$; we notice that $Z \cup \overline{Z} = \wp(X)$ and $Z \cap \overline{Z} = \varnothing$ always hold. So we have found a complement for each subset of X and $(\wp(X), \subseteq)$ is also a complemented lattice.

4.2 Heyting algebras

Heyting algebras have the necessary and sufficient structure to interpret intuitionistic logic. We want now to introduce them and see that they can be enclosed into a category.

Definition 4.10. *A* ***Heyting algebra*** *is a structure* $(X, \leq, \wedge, \vee, 0, 1, \to)$ *where* $(X, \leq, \wedge, \vee, 0, 1)$ *is a lattice with least element* 0 *and greatest element* 1, $\leq$ *is the partial order associated to the lattice as in Proposition (4.1) and*

$$\begin{array}{rccc} \to\colon & X \times X & \longrightarrow & X \\ & (x, y) & \longmapsto & (x \to y) \end{array}$$

is an operator such that $(z \leq x \to y) \Leftrightarrow (z \wedge x \leq y)$.

Lemma 4.1. *Let* $(X, \leq, \wedge, \vee, 0, 1, \to)$ *be a Heyting algebra. The following properties hold for every* $x, y, z \in X$*::*

1. *if* $y \leq z$ *then* $x \wedge y \leq x \wedge z$*;*
2. *if* $y \leq z$ *then* $x \vee y \leq x \vee z$*;*
3. $x \wedge (x \to y) \leq y$*;*
4. *if* $y \leq z$ *then* $x \to y \leq x \to z$*.*

Proof. Let $x, y, z \in X$. Then:

1. $x \wedge y \leq y \leq z$ and $x \wedge y \leq x$. Then $x \wedge y \leq x \wedge z$;
2. $y \leq z \leq z \vee x$ and $x \leq x \vee z$. Then $x \vee y \leq x \vee z$;
3. by definition of $\to$ we have that $(x \to y) \wedge x \leq y$ is equivalent to $(x \to y) \leq x \to y$, which holds by reflexivity;
4. from the previous point $x \wedge (x \to y) \leq y$, but we also know that $y \leq z$, so $x \wedge (x \to y) \leq z$, or equivalently $(x \to y) \wedge x \leq z$, hence $(x \to y) \leq x \to z$.

□

Proposition 4.5. *Let* $(X, \leq, \wedge, \vee, 0, 1, \to)$ *be a Heyting algebra. The distributivity holds, thus for every* $x, y, z \in X$*:*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Proof. It suffices to show one of the two equalities, since the other can be proved in a similar way. Let's consider the first equality of the statement.
By definition of $\rightarrow$: $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$ if and only if $y \vee z \leq x \rightarrow ((x \wedge y) \vee (x \wedge z))$. The last inequality holds if and only if $y \leq x \rightarrow ((x \wedge y) \vee (x \wedge z))$ and $z \leq x \rightarrow ((x \wedge y) \vee (x \wedge z))$, and equivalently if and only if $y \wedge x \leq (x \wedge y) \vee (x \wedge z)$ and $z \wedge x \leq (x \wedge y) \vee (x \wedge z)$. Both of them clearly hold by commutativity and properties of $\vee$.

Conversely, by definition of infimum: $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$ if and only if $(x \wedge y) \vee (x \wedge z) \leq x$ and $(x \wedge y) \vee (x \wedge z) \leq (y \vee z)$ both hold. However by definition of supremum $(x \wedge y) \vee (x \wedge z) \leq x$ is equivalent to $x \wedge y \leq x$ and $x \wedge z \leq x$ (which both hold), and by definition of infimum $(x \wedge y) \vee (x \wedge z) \leq (y \vee z)$ is equivalent to $x \wedge y \leq y \vee z$ and $x \wedge z \leq y \vee z$ (which both hold). □

Lemma 4.2. *Let $(X, \leq, \wedge, \vee, 0, 1, \rightarrow)$ be a Heyting algebra, let $(x_i)_{i \in I}, y \in X$. Then:*

$$y \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (y \wedge x_i).$$

Proof. By definition of $\rightarrow$: $y \wedge \bigvee_{i \in I} x_i \leq \bigvee_{i \in I}(y \wedge x_i)$ if and only if $\bigvee_{i \in I} x_i \leq y \rightarrow \bigvee_{i \in I}(y \wedge x_i)$, and this holds if and only if $x_i \leq y \rightarrow \bigvee_{i \in I}(y \wedge x_i)$ for all $i \in I$, which means $x_i \wedge y \leq \bigvee_{i \in I}(y \wedge x_i)$ and this inequality holds by definition of supremum.

Conversely, by definition of infimum: $\bigvee_{i \in I}(y \wedge x_i) \leq y \wedge \bigvee_{i \in I} x_i$ if and only if both $\bigvee_{i \in I}(y \wedge x_i) \leq y$ and $\bigvee_{i \in I}(y \wedge x_i) \leq \bigvee_{i \in I} x_i$ hold. In fact they are true: the first holds since $y \wedge x_i \leq y$ for all $i \in I$ by definition of infimum, the second holds since $y \wedge x_i \leq x_i$ for all $i \in I$ by definition of infimum. □

Example 4.2. Let (X, τ) be a topological space, where $\tau := \{O | O \text{ is an open set }\}$ is its topology. $((X, \tau), \subseteq)$ is a Heyting algebra.

It is straightforward to show that $\subseteq$ is an order on τ. Given $O_1, O_2 \in \tau$, their infimum is given by $O_1 \cap O_2$. Indeed, $O_1 \cap O_2 \subseteq O_1$ and $O_1 \cap O_2 \subseteq O_2$, and if $O' \in \tau$ is such that $O' \subseteq O_1$ and $O' \subseteq O_2$ then $O' \subseteq O_1 \cap O_2$. Similarly, the supremum of 0_1 and O_2 is given by $O_1 \cup O_2$.
Since $\varnothing \subseteq O$ and $O \subseteq X$ for every $O \in \tau$, the least and greatest elements of $((X, \tau), \subseteq)$ are $\varnothing$ and X respectively.
Let $O_1, O_2 \in \tau$, we define:

$$O_1 \rightarrow O_2 := \bigcup \{O \in \tau | O \cap O_1 \subseteq O_2\}$$

so it is straightforward that $O \subseteq O_1 \longrightarrow O_2 \Leftrightarrow O \cup O_1 \subseteq O_2$.

Lemma 4.3. *Let $(X, \leq, \wedge, \vee, 0, 1, \rightarrow)$ and $(X', \leq, \wedge, \vee, 0, 1, \rightarrow)$ be Heyting algebras and let $\varphi : X \longrightarrow X'$ be a $\wedge$-preserving function between them. Then φ is a monotone function.*

Proof. Let $x, y \in X$ such that $x \leq y$. Then: $\varphi(x) = \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$, thus $\varphi(x) \leq \varphi(y)$. □

Definition 4.11. *Given the Heyting algebras $(X, \leq, \wedge, \vee, 0, 1, \rightarrow)$ and $(X', \leq, \wedge, \vee, 0, 1, \rightarrow)$, a* ***homomorphism of Heyting algebras*** *between them is a function $\varphi : X \longrightarrow X'$ such that for every $x, y \in X$:*

- $\varphi(0) = 0$;
- $\varphi(1) = 1$;
- $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$;
- $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$;
- $\varphi(x \rightarrow y) = \varphi(x) \rightarrow \varphi(y)$.

Definition 4.12. *The category* ***Ha*** *is the subcategory of* ***Set*** *whose objects are Heyting algebras and whose maps are the homomorphisms of Heyting algebras.*

Ha is indeed a category. If $\varphi : X \longrightarrow X'$ and $\psi : X' \longrightarrow X''$ are homomorphisms of Heyting algebras, their composition $\psi \circ \varphi$ is as well:

- $\psi \circ \varphi(0) = \psi(\varphi(0)) = \psi(0) = 0$;
- $\psi \circ \varphi(1) = \psi(\varphi(1)) = \psi(1) = 1$;
- $\psi \circ \varphi(x \wedge y) = \psi(\varphi(x \wedge y) = \psi(\varphi(x) \wedge \varphi(y)) = \psi(\varphi(x)) \wedge \psi(\varphi(y)) = \psi \circ \varphi(x) \wedge \psi \circ \varphi(y)$;
- $\psi \circ \varphi(x \vee y) = \psi(\varphi(x \vee y) = \psi(\varphi(x) \vee \varphi(y)) = \psi(\varphi(x)) \vee \psi(\varphi(y)) = \psi \circ \varphi(x) \vee \psi \circ \varphi(y)$;
- $\psi \circ \varphi(x \to y) = \psi(\varphi(x \to y) = \psi(\varphi(x) \to \varphi(y)) = \psi(\varphi(x)) \to \psi(\varphi(y)) = \psi \circ \varphi(x) \to \psi \circ \varphi(y)$.

The identity map of a Heyting algebra is trivially a homomorphism. Being functions between sets, homorphisms of Heyting algebras respect associativity and the identity.

4.3 Boolean algebras

Boolean algebras are a particular kind of Heyting algebras and have the necessary and sufficient structure to interpret classical logic. We want now to introduce them and to enclose them into a category.

Definition 4.13. *A **Boolean algebra** $(X, \leq, \wedge, \vee, 0, 1)$ is a complemented distributive lattice ($\leq$ is the partial order associated to the lattice as in Proposition (4.1)).*

A Boolean algebra is a particular case of Heyting algebra. In order to show this, it is sufficient to prove that it is possible to define an operation $\to$ such that it behaves in the same way as it does in a Heyting algebra.

Theorem 4.1. *Let $(X, \leq, \wedge, \vee, 0, 1)$ be a Boolean algebra. Then it is also a Heyting algebra.*

Proof. It suffices to show that it is possible to define an operation $\to: X \times X \longrightarrow X$, $(x, y) \longmapsto x \to y$ such that $z \leq x \to y$ if and only if $z \wedge x \leq y$, for every $x, y, z \in X$. We define $x \longrightarrow y := \overline{x} \vee y$ for every $x, y \in X$, where with $\overline{x}$ we indicate the complement of x. This is exactly what we need:

- let's suppose $z \in X$ such that $z \leq x \to y$, then $z \leq \overline{x} \vee y$. Hence $z \wedge x \leq (\overline{x} \vee y) \wedge x \leq (\overline{x} \wedge x) \vee (y \wedge x) = 0 \vee (x \wedge y) = x \wedge y \leq y$;
- let's suppose $z \in X$ such that $z \wedge x \leq y$, then $\overline{x} \vee (z \wedge x) \leq \overline{x} \vee y$. By distributivity: $(\overline{x} \vee z) \wedge (\overline{x} \vee x) \leq \overline{x} \vee y$, so $(\overline{x} \vee z) \wedge 1 \leq \overline{x} \vee y$, which is $\overline{x} \vee z \leq \overline{x} \vee y$. By the definition of supremum: $z \leq \overline{x} \vee y$.

□

We denote $\neg x := x \to 0$, where $\to$ is defined as in the previous Theorem, thus $\neg x = \overline{x} \vee 0 = \overline{x}$, so $\neg x$ coincides with the complement of x. We characterize clearly the connection between Heyting and Boolean algebras thorugh the following results.

Proposition 4.6. *$(X, \leq, \wedge, \vee, 0, 1)$ is a Boolean algebra if and only if it is a Heyting algebra and for every $x \in X$ one has $x \vee \neg x = 1$.*

Proof. ($\Rightarrow$) Let $(X, \leq, \wedge, \vee, 0, 1)$ be a Boolean algebra. Then it is a Heyting algebra by Theorem (4.1). We noticed some lines above that in a Boolean algebra $\neg x$ coincides with the complement $\overline{x}$, hence by definition of complement $x \vee \neg x = 1$ for all $x \in X$.

($\Leftarrow$) Let $(X, \leq, \wedge, \vee, 0, 1)$ be a Heyting algebra where $x \vee \neg x = 1$ for all $x \in X$. $(X, \leq, \wedge, \vee, 0, 1)$ is a distributive lattice by Proposition (4.5). It only remains to show that every element $x \in X$ has a complement $\overline{x} \in X$. Let $x \in X$, then $\neg x$ is its complement:

- $x \vee \neg x = 1$ by hypothesis;
- $x \wedge \neg x = 0$, indeed $0 \leq x \wedge \neg x$ by definition of O in a lattice and $x \wedge \neg x \leq 0$ by definition of $\neg$ and Proposition (4.1).

□

Corollary 4.1. *Let* $(X, \leq, \wedge, \vee, 0, 1, \rightarrow)$ *be a Heyting algebra such that* $\neg\neg x = x$ *for every* $x \in X$. *Then it is a Boolean algebra.*

Proof. By Proposition (4.6) it suffices to show that $x \vee \neg x = 1$ for all $x \in X$.
First of all, we notice that $\neg 0 = 1$, since $\neg 0 \leq 1$ by definition of 1 in a lattice and $1 \leq 0$ by definition of $\rightarrow$ (for: $1 \leq \neg 0 = 0 \rightarrow 0 \Leftrightarrow 1 \wedge 0 \leq 0 \Leftrightarrow 0 \leq 0$). We recall that $x \wedge \neg x = 0$ for all $x \in X$ by Proposition (4.1). Considering the hypothesis $x = \neg\neg x$, it suffices to show that $x \vee \neg x = \neg\neg x \vee \neg x = \neg(\neg x \wedge x)$ for all $x \in X$.

- For all $z, y \in X$, given that $z = \neg\neg z = \neg z \rightarrow 0$, we have:

$$z \wedge y \leq z \Leftrightarrow (z \wedge y) \wedge \neg z \leq 0 \Leftrightarrow \neg z \wedge (z \wedge y) \leq 0 \Leftrightarrow \neg z \leq (z \wedge y) \rightarrow 0 \Leftrightarrow \neg z \leq \neg(z \wedge y)$$

  In particular, if $z \equiv x$, $y \equiv \neg x$ then $x \wedge \neg x \leq x \Rightarrow \neg x \leq \neg(x \wedge \neg x)$ and if $z \equiv \neg x$, $y \equiv x$ then $\neg x \wedge x \leq \neg x \Rightarrow \neg\neg x \leq \neg(\neg x \wedge x)$. By definition of join we infer that $\neg x \vee \neg\neg x \leq \neg(x \wedge \neg x)$.

- For all $z, y \in X$, given that $(z \vee y) = \neg\neg(z \vee y)$, we have:

$$z \leq z \vee y \Leftrightarrow z \wedge \neg(z \vee y) \leq 0 \Leftrightarrow \neg(z \vee y) \wedge z \leq 0 \Leftrightarrow \neg(z \vee y) \leq \neg z$$

  In particular, if $z \equiv \neg x$, $y \equiv \neg\neg x$ then $\neg x \leq \neg x \vee \neg\neg x \Rightarrow \neg(\neg x \vee \neg\neg x) \leq \neg\neg x = x$ and if $z \equiv \neg\neg x$, $y \equiv \neg x$ then $\neg\neg x \leq \neg\neg x \vee \neg x \Rightarrow \neg(\neg\neg x \vee \neg x) \leq \neg\neg\neg x = \neg x)$. Hence by definition of meet $\neg(\neg x \vee \neg\neg x) \leq x \wedge \neg x$, thus $\neg(x \wedge \neg x) \leq \neg\neg(\neg x \vee \neg\neg x) = \neg x \vee \neg\neg x$.

□

In a nutshell, a Boolean algebra happens to be a distributive lattice with a well-define operation of **negation** $\neg$. We can as well define the operation $\neg$ in a Heyting algebra, setting $\neg x = x \rightarrow 0$. However, the structure of a Boolean algebra forces more properties on $\neg$ than a Heyting algebra does. Crucially, in a Boolean algebra the operation $\neg$ turns out to be exactly the complement, which is false in general in a Heyting algebra. In the following Lemma we collect some properties of $\neg$ that hold in Heyting algebras.

Lemma 4.4. *Let* $(X, \leq, \wedge, \vee, 0, 1, \rightarrow)$ *be a Heyting algebra. Then for every* $x, y \in X$*:*

1. $x \wedge \neg x \leq 0$;
2. *if* $x \leq y$ *then* $\neg y \leq \neg x$;
3. $x \leq \neg\neg x$;
4. $\neg(x \vee y) \leq \neg x \wedge \neg y$;
5. $\neg x \vee \neg y \leq \neg(x \wedge y)$.

Proof.
1. By proposition (4.1) we know that $x \wedge (x \rightarrow y) \leq y$. In particular, with $y \equiv 0$, we immediately deduce that $x \wedge (x \rightarrow 0) = x \wedge \neg x \leq 0$.
2. If $x \leq y$ then $x \wedge \neg y \leq y \wedge \neg y$ and $y \wedge \neg y \leq 0$, because of the previous point. We infer immediately then: $x \wedge \neg y \leq 0 \Rightarrow \neg y \leq x \rightarrow 0 \Rightarrow \neg y \leq \neg x$.
3. By the first point of this Lemma and by definition of $\rightarrow$ we have: $x \wedge \neg x \leq 0 \Leftrightarrow x \wedge (x \rightarrow 0) \leq 0 \Leftrightarrow x \leq (x \rightarrow 0) \rightarrow 0 = \neg\neg x$.
4. By definition of join: $x \leq x \vee y$ and $y \leq x \vee y$, thus by the second point of this Lemma $\neg(x \vee y) \leq \neg x$ and $\neg(x \vee y) \leq \neg y$. By definition of meet: $\neg(x \vee y) \leq \neg x \wedge \neg y$.
5. By definition of meet: $x \wedge y \leq x$ and $x \wedge y \leq y$, thus by the second point of this Lemma $\neg x \leq \neg(x \wedge y)$ and $\neg y) \leq \neg(x \wedge y)$. By definition of join: $\neg x \vee \neg y \leq \neg(x \wedge y)$.

□

Proposition 4.7. *Let $(X, \leq, \wedge, \vee, 0, 1, \rightarrow)$ be a Heyting algebra. The following conditions are equivalent for all $x \in X$:*

(i) $\neg\neg x \leq x$;

(ii) $1 \leq x \vee \neg x$;

Proof. $(i) \Rightarrow (ii)$ Let's suppose that $\neg\neg x \leq x$ for all $x \in X$. By the first point of Lemma (4.4) applied to $\neg x$, we get: $\neg x \wedge \neg\neg x \leq 0$. Besides, by point 4 of Lemma (4.4) with $y \equiv \neg x$: $\neg(x \vee \neg x) \leq \neg x \wedge \neg\neg x$. Hence $\neg(x \vee \neg x) \leq 0 \Rightarrow 1 \wedge \neg(x \vee \neg x) \leq 0 \Rightarrow 1 \leq \neg(x \vee \neg x) \rightarrow 0 \Rightarrow 1 \leq \neg\neg(x \vee \neg x)$. Applying the hypothesis to $(x \vee \neg x)$ we infer $\neg\neg(x \vee \neg x) \leq (x \vee \neg x)$, and we can conclude $1 \leq (x \vee \neg x)$.

$(ii) \Rightarrow (i)$ Let's suppose that $1 \leq x \vee \neg x$ for all $x \in X$. Since $\neg\neg x \leq 1 \wedge \neg\neg x$, we deduce $\neg\neg x \leq \neg\neg x \wedge (x \vee \neg x) = (\neg\neg x \wedge x) \vee (\neg\neg x \wedge \neg x)$. By the first point of Lemma (4.4): $\neg\neg x \wedge \neg x \leq 0$, and by definition of meet: $\neg\neg x \wedge x \leq x$. So concluding: $\neg\neg x \leq (\neg\neg x \wedge x) \vee (\neg\neg x \wedge \neg x) \leq x \vee 0 = x$. □

Definition 4.14. *Given the Boolean algebras $(X, \leq, \wedge, \vee, 0, 1)$ and $(X', \leq, \wedge, \vee, 0, 1)$, a homomorphism of Boolean algebras between them is a function $\varphi : X \longrightarrow X'$ such that for every $x, y \in X$:*

- $\varphi(0) = 0$;
- $\varphi(1) = 1$;
- $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$;
- $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$;
- $\varphi(x \rightarrow y) = \varphi(x) \rightarrow \varphi(y)$.

Lemma 4.5. *Let $(X, \leq, \wedge, \vee, 0, 1)$, $(X', \leq, \wedge, \vee, 0', 1')$ be Boolean algebras and let $f : X \longrightarrow X'$ be a function such that $f(0) = 0'$, $f(1) = 1'$, $(x \wedge y) = f(x) \wedge f(y)$ and $(x \vee y) = f(x) \vee f(y)$ for all $x, y \in X$. Then f is a homomorphism of Heyting algebras.*

Proof. It suffices to show that $f(x \rightarrow y) = f(x) \rightarrow f(y)$ for every $x, y \in X$. We recall that in a Boolean algebra $x \rightarrow y \equiv \overline{x} \vee y$ where $\overline{x}$ is the complement of x, hence $x \rightarrow y \equiv \neg x \vee y$. It follows that $f(x \rightarrow y) = f(\neg x \vee y) = f(\neg x) \vee f(y)$ and $f(x) \rightarrow f(y) = \neg f(x) \vee f(y)$, so it suffices to show that $f(\neg x) = \neg f(x)$ for every $x \in X$. This holds since $f(x) \vee f(\neg x) = f(x \vee \neg x) = f(1) = 1' = f(x) \vee \neg f(x)$ and $f(x) \wedge f(\neg x) = f(x \wedge \neg x) = f(0) = 0' = f(x) \wedge \neg f(x)$, thus both $\neg f(x)$ and $f(\neg x)$ are complements of $f(x)$ and they must be equal, due to the uniqueness of the complement. □

Lemma 4.6. *Let $(X, \leq, \wedge, \vee, 0, 1)$, $(X', \leq, \wedge, \vee, 0', 1')$ be Boolean algebras and let $f : X \longrightarrow X'$ be a function such that $f(0) = 0'$, $f(1) = 1'$, $f(\neg x) = \neg f(x)$ and $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in X$. Then f is a homomorphism of Boolean algebras.*

Proof. Let $x, y \in X$. We recall that in a Boolean algebra $x \rightarrow y \equiv \overline{x} \vee y$ where $\overline{x}$ is the complement of x, then $x \rightarrow y \equiv \neg x \vee y$. Therefore:

$$f(x \rightarrow y) = f(\neg x \vee y) = f(\neg x) \vee f(y) = \neg f(x) \vee f(y) = f(x) \longrightarrow f(y).$$

Since $\neg(a \vee b) = \neg a \wedge \neg b$ and $a = \neg\neg a$ for every a, b in a Boolean algebra, we infer:

$$f(x \wedge y) = f(\neg(\neg x \vee \neg y)) = \neg(f(\neg x) \vee f(\neg y)) = \neg(\neg f(x) \vee \neg f(y)) = \neg\neg f(x) \wedge \neg\neg f(y) = f(x) \wedge f(y).$$

□

Definition 4.15. *The category **Ba** is the subcategory of **Set** whose objects are Boolean algebras and whose maps are the homomorphisms of Boolean algebras.*

The fact that **Ba** is indeed a category follows directly from the fact that **Ha** is.

4.4 Lindenbaum algebras

Let $\mathcal{T}$ be a logical theory. The **Lindenbaum** algebra of $\mathcal{T}$ is obtained as the quotient of the sentences of the theory, under the equivalence relation of logical equivalence (i.e., two sentences are equivalent if the theory $\mathcal{T}$ proves that each one of them implies the other, a logical deduction system being set). This algebra is named after logicians Adolf Lindenbaum and Alfred Tarski: it was first introduced by Tarski in 1935 as a device to establish correspondence between classical propositional calculus and Boolean algebras.

Let's consider a language $\mathcal{L}$ for the intuitionistic predicative logic IL. Let $Form_{\mathcal{L}}$ be the collection of all $\mathcal{L}$-formulas and $Der(IL)$ be the collection of all the derivations of LI_p in the language $\mathcal{L}$. We write $\varphi \vdash \psi$ if there exists a derivation $D \in Der(IL)$ such that φ appears among its assumptions and ψ is its conclusion.

Lemma 4.7. $(Form_{\mathcal{L}}, \vdash)$ *is a preorder.*

Proof. We need to show that $\vdash$ is a reflexive and transitive relation. $\varphi \vdash \varphi$ holds for every $\varphi \in Form_{\mathcal{L}}$ by definition of derivation. For all $\varphi, \psi, \chi \in Form_{\mathcal{L}}$, if $\varphi \vdash \psi$ and $\psi \vdash \chi$, then we can obtain a derivation of $\varphi \vdash \chi$ by juxtaposing the derivations of $\varphi \vdash \psi$ and $\psi \vdash \chi$. □

We define the equivalence relation $\sim$ on $Form_{\mathcal{L}}$ as follows:

$$\text{for every } \varphi, \psi \in Form_{\mathcal{L}} : \varphi \sim \psi \Leftrightarrow \varphi \vdash \psi \text{ and } \psi \vdash \varphi.$$

This is indeed an equivalence relation, as it is easy to see, and so gives rise to a quotient class for every formula $\varphi \in Form_{\mathcal{L}}$

$$[\varphi] := \{\psi \in Form_{\mathcal{L}} | \varphi \sim \psi\}$$

and to the quotient set

$$\mathfrak{A}_i(\mathcal{L}) := \{[\varphi] | \varphi \in Form_{\mathcal{L}}\}.$$

We define the relation $\leq$ on $\mathfrak{A}_i(\mathcal{L})$ by

$$[\varphi] \leq [\psi] \Leftrightarrow \varphi \vdash \psi.$$

This actually is a well-defined relation on $\mathfrak{A}_i(\mathcal{L})$. If $[\varphi_1] = [\varphi_2]$ and $[\psi_1] = [\psi_2]$ then in particular there are derivations D' and D''' for $\varphi_2 \vdash \varphi_1$ and $\psi_1 \vdash \psi_2$. Let's suppose that D'' is a derivation for $\varphi_1 \dashv \psi_1$, then the juxtaposition $D' \cup D'' \cup D'''$ is a derivation for $\varphi_2 \vdash \psi_2$. Similarly, from a derivation for $\varphi_2 \vdash \psi_2$ we can obtain a derivation for $\varphi_1 \vdash \psi_1$.

Lemma 4.8. $(\mathfrak{A}_i(\mathcal{L}), \leq)$ *is an order.*

Proof. We already know that $(Form_{\mathcal{L}}, \vdash)$ is a preorder. It only suffices to show that $\leq$ is antisymmetric in $\mathfrak{A}_i(\mathcal{L})$, and this is trivially true since $[\varphi] \leq [\psi]$ and $[\psi] \leq [\varphi]$ if and only if $\varphi \vdash \psi$ and $\psi \dashv \varphi$ if and only if $\varphi \sim \psi$ if and only if $[\varphi] = [\psi]$. □

Theorem 4.2. *Let $\mathcal{L}$ be a language for the intuitionistic predicative logic IL and let $\mathfrak{A}_i(\mathcal{L})$ be the corresponding Lindenbaum algebra. Then $(\mathfrak{A}_i(\mathcal{L}), \leq)$ is a Heyting algebra.*

Proof. $(\mathfrak{A}_i(\mathcal{L}), \leq)$ is a lattice where for every $\varphi, \psi \in Form_{\mathcal{L}}$ their join and meet are given by

$$[\varphi] \vee [\psi] = [\varphi \vee \psi] \text{ and } [\varphi] \wedge [\psi] = [\varphi \wedge \psi].$$

Indeed, considering the join:

- $[\varphi] \leq [\varphi \vee \psi]$ and $[\psi] \leq [\varphi \vee \psi]$, since

$$\frac{\overline{\varphi \vdash \varphi}}{\varphi \vdash \varphi \vee \psi} \quad \text{and} \quad \frac{\overline{\psi \vdash \psi}}{\psi \vdash \varphi \vee \psi}$$

- if $[\varphi] \leq [\chi]$ and $[\psi] \leq [\chi]$ then $[\varphi \vee \psi] \leq [\chi]$, since

$$\dfrac{\overline{\varphi \vdash \chi} \qquad \overline{\psi \vdash \chi}}{\varphi \vee \psi \vdash \chi}$$

and considering the meet:

- $[\varphi \wedge \psi] \leq [\varphi]$ and $[\varphi \wedge \psi] \leq [\psi]$, since

$$\dfrac{\overline{\varphi, \psi \vdash \varphi}}{\varphi \wedge \psi \vdash \varphi} \quad \text{and} \quad \dfrac{\overline{\varphi, \psi \vdash \psi}}{\varphi \wedge \psi \vdash \psi}$$

- if $[\chi] \leq [\varphi]$ and $[\chi] \leq [\psi]$ then $[\chi] \leq [\varphi \wedge \psi]$, since

$$\dfrac{\overline{\chi \vdash \varphi} \qquad \overline{\chi \vdash \psi}}{\chi \vdash \varphi \wedge \psi}$$

The lattice $(\mathfrak{A}_i(\mathcal{L}), \leq)$ has least element $0 \equiv [\bot]$ and greatest element $1 \equiv [taut]$, where $taut$ is any tautology. For any formula $\varphi \in Form_{\mathcal{L}}$: $\bot \vdash \varphi$ holds and so $[\bot] \leq [\varphi]$, $\varphi \vdash taut$ holds and so $[\varphi] \leq [taut]$.
We define in $(\mathfrak{A}_i(\mathcal{L}), \leq)$ an operation $\rightarrow$ by

$$\begin{array}{rccc} \rightarrow: & (\mathfrak{A}_i(\mathcal{L}), \leq) \times (\mathfrak{A}_i(\mathcal{L}), \leq) & \longrightarrow & (\mathfrak{A}_i(\mathcal{L}), \leq) \\ & ([\varphi], [\psi]) & \longmapsto & [\varphi \rightarrow \psi] \end{array}$$

It is well-defined and such that $[\varphi] \leq [\psi \rightarrow \chi] \Rightarrow [\varphi] \wedge [\psi] = [\varphi \wedge \psi] \leq [\chi]$, indeed:

$$\dfrac{\dfrac{\overline{\varphi \wedge \psi \vdash \chi}}{\varphi, \psi \vdash \chi}}{\varphi \vdash \psi \rightarrow \chi} \quad \text{and} \quad \dfrac{\dfrac{\overline{\varphi \vdash \psi \rightarrow \chi}}{\varphi, \psi \vdash \chi}}{\varphi \wedge \psi \vdash \chi}$$

□

Proceeding in the same way as before, but considering a language for the classical predicative logic CL, we get to the same result for classical logic.

Theorem 4.3. *Let $\mathcal{L}$ be a language for the classical predicative logic CL and let $\mathfrak{A}_c(\mathcal{L})$ be the corresponding Lindenbaum algebra. Then $(\mathfrak{A}_c(\mathcal{L}), \leq)$ is a Boolean algebra.*

Proof. Thanks to Theorem (4.2) and to Proposition (4.6), it suffices to show that $[\varphi] \vee \neg[\varphi] = 1$ for every $[\varphi] \in Form_{\mathcal{L}}$. However we know from Theorem (4.2) that $1 = [taut]$ and that $\neg[\varphi] = [\varphi] \rightarrow 0 = [\varphi] \rightarrow [\bot] = [\varphi \rightarrow \bot] = [\neg\varphi]$, and so $[\varphi] \vee \neg[\varphi] = [\varphi] \vee [\neg\varphi] = [\varphi \vee \neg\varphi]$. Therefore we need to show that $[\varphi \vee \neg\varphi] = [taut]$, but this surely holds since $\varphi \vee \neg\varphi$ is a classical tautology for every $\varphi \in Form_{\mathcal{L}}$. □

4.5 The Dedekind-MacNeille completion

Let X be a given preorder[1]. A **completion of the preorder** X is a complete lattice $\widetilde{X}$ with an order-embedding of X into $\widetilde{X}$. Since $\widetilde{X}$ is a complete lattice, every subset of elements of $\widetilde{X}$ has infimum and supremum, and since $X \longrightarrow \widetilde{X}$ is an order-embedding, distinct elements of X must be mapped to distinct elements of $\widetilde{X}$ and each pair of elements in X has the same ordering in $\widetilde{X}$ as they do in X.

[1] See Chapter Preorders for a rigorous definition.

The poset X may have several different completions. Among all those, the **Dedekind–MacNeille completion** (also called **completion by cuts** or **normal completion**) is the smallest complete lattice that contains the poset X itself. It is named after Holbrook Mann MacNeille (1907-1973), who first defined and formalized this completion, and after Richard Dedekind (1831-1916), because this completion is a generalization of the Dedekind cuts used by Dedekind to construct the real numbers from the rational numbers.

Definition 4.16. *Let $(P, \leq)$ be a preorder. For every $V \subseteq P$, we indicate the upper bounds and the lower bounds of V in P by:*

$$V^{\uparrow} := \{z \in P | \forall x \in V(x \leq z)\} \qquad V^{\downarrow} := \{z \in P | \forall x \in V(z \leq x)\}$$

*The **Dedekind-MacNeille completion** of the preorder $(P, \leq)$ is the algebra $DM(P)$ defined by:*

$$DM(P) := \{V \subseteq P | V = (V^{\uparrow})^{\downarrow}\}$$

*hence every $V \in DM(P)$ has elements x such that $\forall y \in P(\forall z \in V(z \leq y)) \to x \leq y$. Let $(P, \leq)$ be a preorder. The **Dedekind-MacNeille operator** is given by:*

$$\begin{array}{rccc} DM: & \wp(P) & \longrightarrow & \wp(P) \\ & V & \longmapsto & DM(V) = \{x \in V | \forall y \in P(\forall z \in V(z \leq y)) \to x \leq y\} \end{array}$$

The Dedekind-MacNeille operator is a particular kind of operator, namely a closure operator.

Definition 4.17. *Let X be a set. The operator $T : \wp(X) \longrightarrow \wp(X)$ is a **closure operator** if for every $U, V \in \wp(X)$:*

(i) T is reflexive: $U \subseteq T(U)$;

(ii) T is idempotent: $T(T(U)) \subseteq U$;

(iii) T is monotone: $U \subseteq V \to T(U) \subseteq T(V)$.

Lemma 4.9. *The Dedekind-MacNeille operator is a closure operator.*

Proof. Let's consider the Dedekind-MacNeille operator for a given preorder P:

$$DM : V \longmapsto DM(V) = \{x \in V | \forall y \in P(\forall z \in V(z \leq y)) \to x \leq y\}$$

where $V \in \wp(P)$.

(i) DM is reflexive: Let $V \in \wp(P)$. Trivially for every $x \in V$ the condition $\forall y \in P(\forall z \in V(z \leq y)) \to x \leq y$ holds. Hence $V \subseteq DM(V)$.

(ii) DM is idempotent: Let $V \in \wp(P)$ and $x \in DM(DM(V))$. Then $\forall y \in P(\forall z \in DM(V)(z \leq y)) \to x \leq y$ holds, and since $V \subseteq DM(V)$ also $\forall y \in P(\forall z \in V(z \leq y)) \to x \leq y$ holds, and so $x \in DM(V)$. Hence $DM(DM(V)) \subseteq V$.

(iii) DM is monotone: Let $U, V \in \wp(P)$ such that $U \subseteq V$. As it is easy to show, both $()^{\uparrow}$ and $()^{\downarrow}$ are antimonotone, thus $U \subseteq V \Rightarrow V^{\uparrow} \subseteq U^{\uparrow} \Rightarrow (U^{\uparrow})^{\downarrow} \subseteq (V^{\uparrow})^{\downarrow}$, hence DM is monotone.

□

Theorem 4.4. *Let P be a preorder. The Dedekind-MacNeille completion $DM(P)$ of P is still a preorder, with respect to the set inclusion. If P is a Heyting (respectively Boolean) algebra, then $DM(P)$ is a complete Heyting (respectively Boolean) algebra.*

Proof. The first claim is trivially true. Let's suppose P to be a Heyting algebra and let $U, V \in \wp(P)$.

- $U \wedge_{DM(P)} V \equiv U \cap U$. Indeed: $DM(U \cup V) \subseteq DM(U) = U$ and $DM(U \cap V) \subseteq DM(V) = V$ by monotonicity (Lemma (4.9)), thus $DM(U \cap V) \subseteq U \cap V$. By reflexivity (Lemma (4.9)) $U \cap V \subseteq DM(U \cap V)$. Concluding: $U \cap V = DM(U \cap V)$. (More generally $\bigwedge_{i \in I} V_i \equiv \bigcap_{i \in I} V_i$.)

- $U \vee_{DM(P)} V \equiv DM(U \cup V)$. Indeed: by monotonicity (Lemma (4.9)) $U = DM(U) \subseteq DM(U \cup V)$ and $V = DM(V) \subseteq DM(U \cup V)$. If $W \in DM(P)$ such that $U \subseteq W$ and $V \subseteq W$ then $U \cup V \subseteq W$ and by monotonicity (Lemma (4.9)) $DM(U \cup V) \subseteq DM(W) = W$. (More generally $\bigvee_{i \in I} V_i \equiv DM(\bigcup_{i \in I} V_i)$.)
- Trivially, $1_{DM(P)} \equiv P$ and $0_{DM(P)} \equiv \{0_P\}$.
- $U \rightarrow_{DM(P)} V \equiv \{p \in P | \forall u \in U(p \wedge u \in V)\}$. Indeed: for every $W \in DM(P)$, $W \subseteq U \rightarrow V \Leftrightarrow W \cap U \subseteq V$. In order to show that, we previously notice that, given $W \in DM(P)$:

$$W \subseteq U \rightarrow V \overset{(*)}{\Leftrightarrow} W \cap U \subseteq V$$

since:

($\Rightarrow$) Let $z \in W \cap U$. In particular $z \in W$ and since $W \subseteq U \rightarrow V$, then $z \in U \rightarrow V$. Also $z \in U$, thus $z \wedge z = z \in V$.

($\Leftarrow$) Let $w \in W$ and $u \in U$. By definition of the Dedekind-MacNeille operator, $w \wedge u \in W$ and $w \wedge u \in U$. Hence $w \wedge u \in W \cap U$, and since $W \cap U \subseteq V$ then $w \wedge u \in V$. By definition, $w \in U \rightarrow V$.

It remains to show that $DM(U \rightarrow V) = U \rightarrow V$, and by reflexivity (Lemma (4.9)) it suffices to show that $DM(U \rightarrow V) \subseteq U \rightarrow V$. By the idempotence (Lemma (4.9)) $DM(U \rightarrow V) \in DM(P)$, and due to $(*)$: $DM(U \rightarrow V) \subseteq U \rightarrow V \Leftrightarrow DM(U \rightarrow V) \cap U \subseteq V$. Thus, we only have to show that $DM(U \rightarrow V) \cap U \subseteq V$:
Let $z \in DM(U \rightarrow V) \cap U$. If $k \in V^{\uparrow}$, let's take into consideration $z \rightarrow k$: for every $z' \in U \rightarrow V$, $z' \wedge z \in V$ because $z \in U$, thus $z' \wedge z \leq k$ because $k \in V^{\uparrow}$, hence $z' \leq z \rightarrow k$, and concluding $z \rightarrow k \in (U \rightarrow V)^{\uparrow}$. Due to the fact that $z \in DM(U \rightarrow V)$ we infer that $z \leq z \rightarrow k$, thus $z = z \wedge z \leq k$. Concluding: $z \in DM(V) = V$.

Let's suppose P to be a Boolean algebra. By Proposition (4.6) and by the previous part of the proof, it remains to show that $DM(V \cup \neg V) = P$ for every $V \in DM(P)$. Equivalently, let's consider $z \in (V \cup \neg V)^{\uparrow}$ and let's show that $z = 1_P$:
Let $v \in V$. Then $v \leq z$ and so $\neg z \leq \neg v$. It follows that

$$\neg z \in \{\neg v | v \in V\}^{\uparrow}$$

We notice also that

$$\neg V = V \rightarrow O_{DM(P)} = \{p \in P | \forall v \in V(p \wedge v \leq 0_P)\} = \{p \in P | \forall v \in V(p \leq \neg v)\} = \{\neg v | v \in V\}^{\uparrow}$$

We deduce that $\neg z \in \neg V$, and since $z \in (\neg V)^{\uparrow}$ we have that $\neg z \leq z$, thus $1_P = z \wedge \neg z \leq z$, hence $z = 1_P$. ☐

Lemma 4.10. *Let P be a preorder and let $z \in P$. Let's define*

$$\downarrow z := \{x \in P | x \leq z\}$$

Then:

$$\downarrow z = DM(\{z\})$$

where DM is the Dedekind-MacNeille operator.

Proof. Let $z \in P$ and $V := \{z\}$. By Definition (4.16): $DM(\{z\}) = DM(V) = \{x \in P | \forall y \in P(\forall w \in V(w \leq y)) \rightarrow x \leq y\} = \{x \in P | \forall y \in P(z \leq y) \rightarrow x \leq y\}$. We need to show that $\downarrow z = DM(\{z\})$:

($\subseteq$) Let $x \in \downarrow z$, so $x \leq z$, and let $y \in P$ such that $z \leq y$. From transitivity of the relation $\leq$ follows that $x \leq y$, thus $x \in DM(V)$.

($\supseteq$) Let $x \in DM(V)$. Then for every y such that $z \leq y$, we know that $x \leq y$. In particular, let's take $y \equiv z$. Since trivially $z \leq z$ because of reflexivity of the relation $\leq$, we gather that $x \leq z$, hence $x \in \downarrow z$. ☐

Lemma 4.11. *Let P be a preorder and $V \in \wp(P)$, $z \in P$. Then $\downarrow z = \bigvee_{x \in V} \downarrow x$ if and only if $\bigvee_{x \in V} x$ exists in P and $z = \bigvee_{x \in V} x$.*

Proof. We use the notation $\bigvee V \equiv \bigvee_{x \in V} x$.

($\Rightarrow$) Let $\downarrow z = \bigvee_{x \in V} \downarrow x$. Then $\downarrow x \subseteq \downarrow z$ for every $x \in V$, so $x \leq z$ for every $x \in V$ as well. Let y be such that $x \leq y$ for every $x \in V$. Since $z \in \bigvee_{x \in V} \downarrow x = DM(V)$, it follows that $z \leq y$. Concluding: $z = \bigvee V$.

($\Leftarrow$) Let's suppose that $z = \bigvee V$. It follows that $x \leq z$ for every $x \in V$, hence $V \subseteq \downarrow z$. Being DM monotone: $DM(V) \subseteq DM(\downarrow z)$. In addition: $M(\downarrow z) = M(M(\downarrow z)) \subseteq M(\{z\}) = \downarrow z$, thanks to Lemma (4.10). Therefore: $DM(V) \subseteq \downarrow z$. Since $DM(V)$ is complete, we have that $z = \bigvee V \in DM(V)$, and from monotonicity of DM we gather that $\downarrow z = DM(\{z\}) \subseteq DM(DM(V)) = DM(V)$. Concluding: $\downarrow z = DM(V)$. □

Proposition 4.8. *Let A be a Heyting algebra. Then the mapping*

$$\begin{array}{rccc} \downarrow: & A & \longrightarrow & DM(A) \\ & z & \longmapsto & \downarrow z = \{x \in A | x \leq z\} \end{array}$$

is a homomorphism of Heyting algebras preserving existing infinitary meets and joins.

Proof. We denote by 0 and 1 the infimun and the supremum of the Heyting algebra A. To begin with, we notice that the mapping $\downarrow$ is monotone: given $z \leq z'$ in A, we have that $\downarrow z = \{x \in A | x \leq z\} \subseteq \{x \in A | x \leq z'\} = \downarrow z'$. We proceed to show that requirements in Definition (4.11) are met.

- $\downarrow 0 = 0_{DM(A)}$. Indeed:
$$\downarrow 0 = \{x \in A | x \leq 0\} = \{0\} = 0_{DM(A)}$$
- $\downarrow 1 = 1_{DM(A)}$. Indeed:
$$\downarrow 1 = \{x \in A | x \leq 1\} = A = 1_{DM(A)}$$
- $\downarrow (z \wedge z') = \downarrow z \cap \downarrow z'$ for every $z, z' \in A$.

  ($\subseteq$) Being $\downarrow$ monotone and since $z \wedge z' \leq z$, we infer that $\downarrow (z \wedge z') \subseteq \downarrow z$. Similarly $\downarrow (z \wedge z') \subseteq \downarrow z'$. Thus $\downarrow (z \wedge z') \subseteq \downarrow z \cap \downarrow z'$.

  ($\supseteq$) For every $x \in \downarrow z \cap \downarrow z'$, we have that $x \in \downarrow z$ and $x \in \downarrow z'$, so $x \leq z$ and $x \leq z'$. By definition of meet: $x \leq z \wedge z'$, thus $x \in \downarrow (z \wedge z')$. Since this holds for every $x \in \downarrow z \cap \downarrow z'$, we conclude that $\downarrow z \cap \downarrow z' \subseteq \downarrow (z \wedge z')$.
- $\downarrow (z \vee z') = \downarrow z \cup \downarrow z'$ for every $z, z' \in A$.

  ($\subseteq$) For every x such that $\downarrow z \cup \downarrow z' \subseteq \downarrow x$, we have that $\downarrow z \subseteq \downarrow x$ and $\downarrow z \subseteq \downarrow x$, so $z \leq x$ and $z' \leq x$. By definition of join: $z \vee z' \leq x$, thus $\downarrow (z \vee z') \subseteq \downarrow x$. Due to the generality of x, we conclude that $\downarrow (z \vee z') \subseteq \downarrow z \cup \downarrow z'$.

  ($\supseteq$) Being $\downarrow$ monotone and since $z \leq z \vee z'$, we infer that $\downarrow z \subseteq \downarrow (z \vee z')$. Similarly $\downarrow z' \subseteq \downarrow (z \vee z')$. Thus $\downarrow z \cup \downarrow z' \subseteq \downarrow (z \vee z')$.
- $\downarrow (z \to z') = \downarrow z \to \downarrow z'$ for every $z, z' \in A$.

  ($\subseteq$) By definition of implication and since $\downarrow$ preserves meets, it suffices to show that $\downarrow (z \to z') \cap \downarrow z \subseteq \downarrow z'$, which holds if and only if $\downarrow ((z \to z') \wedge z) \subseteq \downarrow z'$, that is if and only if $(z \to z') \wedge z \leq z'$. The last inequality trivially holds by reflexivity of the relation $\leq$, since by definition of implication it is equivalent to $z \to z' \leq z \to z'$.

  ($\supseteq$) For every $x \in \downarrow z \to \downarrow z'$, we have that $\downarrow x \subseteq \downarrow z \to \downarrow z'$, hence $\downarrow x \cap \downarrow z \subseteq \downarrow z'$, that is $\downarrow (x \wedge z) \subseteq \downarrow z'$. Then $x \wedge z \leq z'$, and so by definition of implication: $x \leq z \to z'$. By monotonicity of $\downarrow$: $\downarrow x \subseteq \downarrow (z \wedge z')$.

Furthermore, $\downarrow$ preserves all existing infinitary meets and joins.

- $\downarrow(\bigvee_{x\in X} x) = \bigvee_{x\in X} \downarrow x$ for every $X \subseteq A$. Indeed, by applying Lemma (4.11) with $z \equiv \bigvee_{x\in X} x$, we obtain what is needed.
- $\downarrow(\bigwedge_{x\in X} x) = \bigwedge_{x\in X} \downarrow x$ for every $X \subseteq A$. Indeed, by applying Lemma (4.10) with $z \equiv \bigwedge_{x\in X} x$, we obtain: $\downarrow \bigwedge_{x\in X} x = DM(\bigwedge_{x\in X} x)$. Being DM a closure operator and due to Theorem (4.4):
$$DM(\bigwedge_{x\in X} x) = \bigcap_{x\in X} DM(x) = \bigcap_{x\in X} \downarrow x$$
and since $DM(A)$ is a Heyting algebra we conclude that $\bigcap_{x\in X} \downarrow x = \bigwedge_{x\in X} \downarrow x$.

□

The Dedekind-MacNeille operator can be described in purely category-theoretic terms, via the notion of adjunction. This fact does not come as a surprise: category theory was born as a general language to deal with algebraic structures, and left and right adjoints retrace perfectly the core of many algebraic constructions.

Definition 4.18. *The category **Sup** is defined as follows: its objects are preorders with top element and its morphisms are ordered maps preserving the top and the existing infinitary meets and joins.*

We consider the following embedding functor:

$$\begin{array}{rccc} I_{sup}: & \mathbf{Ha} & \longrightarrow & \mathbf{Sup} \\ & A & \longmapsto & A \\ & (\varphi : A \longrightarrow A') & \longmapsto & (\varphi : A \longrightarrow A') \end{array}$$

and we call $I_{sup}(\mathbf{Ha})$ the full image category of $\mathbf{Ha}$ through I_{sup}. Similarly we can consider the category $\mathbf{cHa}$[2] of complete Heyting algebras and call $I_{sup}(\mathbf{cHa})$ the full image category of $\mathbf{cHa}$ through I_{sup}. There is an embedding functor

$$\begin{array}{rccc} I: & I_{sup}(\mathbf{cHa}) & \longrightarrow & I_{sup}(\mathbf{Ha}) \\ & \hat{A} & \longmapsto & \hat{A} \\ & (\varphi : \hat{A} \longrightarrow \hat{A}') & \longmapsto & (\varphi : \hat{A} \longrightarrow \hat{A}') \end{array}$$

Theorem 4.5. *Let's consider the categories $I_{sup}(\mathbf{Ha})$ and $I_{sup}(\mathbf{cHa})$, the embedding functor I and the Dedekind-MacNeille operator DM. The functor I has a left adjoint, which we call DM, being the Dedekind-MacNeille operator on the objects of $I_{sup}(\mathbf{Ha})$.*

Proof. Let $A \in Ob(I_{sup}(\mathbf{Ha}))$. Then $DM(A) \in Ob(I_{sup}(\mathbf{cHa}))$ by Theorem (4.4). We define the mapping

$$\begin{array}{rccc} \eta_A: & A & \longrightarrow & DM(A) \\ & a & \longmapsto & \downarrow a \end{array}$$

and we need to show that for every $f : A \longrightarrow I(\hat{B}) = \hat{B}$ in $I_{sup}(\mathbf{Ha})$ there exists a unique $\overline{f} : DM(A) \longrightarrow \hat{B}$ in $I_{sup}(\mathbf{cHa})$ such that $I(\overline{f}) \circ \eta_A = f$.

- Let $f : A \longrightarrow \hat{B}$ in $I_{sup}(\mathbf{Ha})$ be given. We define:
$$\begin{array}{rccc} \overline{f}: & DM(A) & \longrightarrow & \hat{B} \\ & X & \longmapsto & \bigvee_{x\in X} f(x) \end{array}$$
This is well-defined since $\hat{B}$ is a complete Heyting algebra, and it is also monotone because f is and trivially:
$$X \subseteq X' \text{ in } DM(A) \Rightarrow \overline{f}(X) = \bigvee_{x\in X} f(x) \leq \bigvee_{x'\in X'} f(x') = \overline{X'}$$

[2]For the sake of clarity we will indicate its objects by $\hat{A}$ with the purpose of discriminating complete and not complete algebras.

- $\overline{f}$ is a map in $I_{sup}(\mathbf{cHa})$: We only need to prove that it preserves the supremum and the existing joins. First of all:

$$\overline{f}(1_{DM(A)}) = \overline{f}(A) = \bigvee_{a \in A} f(a) = \bigvee_{\hat{b} \in \hat{B}} \hat{b} = 1_{\hat{B}}$$

Let's now consider a collection $A_i \subseteq DM(A)$, $i \in I$. We notice that:

$$\overline{f}(\bigvee_{i \in I} A_i) = \overline{f}(DM(\bigcup_{i \in I} A_i)) = \bigvee_{z \in DM(\bigcup_{i \in I} A_i)} f(z)$$

and

$$\bigvee_{i \in I} f(A_i) = \bigvee_{i \in I} \bigvee_{x \in A_i} f(x) = \bigvee_{x \in \bigcup_{i \in I} A_i} f(x)$$

thus it suffices to show that

$$\bigvee_{z \in DM(\bigcup_{i \in I} A_i)} f(z) = \bigvee_{x \in \bigcup_{i \in I} A_i} f(x)$$

($\leq$) Let $z \in DM(\bigcup_{i \in I} A_i)$. Then:

$$\downarrow z \leq \bigvee_{x \in \bigcup_{i \in I} A_i} \downarrow x = DM(\bigcup_{x \in \bigcup_{i \in I} A_i} \downarrow x) = DM(\bigcup_{i \in I} A_i)$$

and so:

$$(\downarrow z) \wedge DM(\bigcup_{i \in I} A_i) = (\downarrow z) \wedge (\bigvee_{x \in \bigcup_{i \in I} A_i} \downarrow x) = \downarrow z$$

By the distributive property:

$$\downarrow z = (\downarrow z) \wedge (\bigvee_{x \in \bigcup_{i \in I} A_i} \downarrow x) = \bigvee_{x \in \bigcup_{i \in I} A_i} (\downarrow z \wedge \downarrow x)$$

However $\downarrow$ is a homomorphism of Heyting algebras, hence:

$$\bigvee_{x \in \bigcup_{i \in I} A_i} (\downarrow z \wedge \downarrow x) = \bigvee_{x \in \bigcup_{i \in I} A_i} \downarrow (z \wedge x)$$

Now we recall that f preserves the existing joins and is monotone, consequently from $z \wedge x \leq z$ and $z \wedge x \leq x$ we gather $f(z \wedge x) \leq f(z)$ and $f(z \wedge x) \leq f(x)$, and so $f(z \wedge x) \leq f(z) \wedge f(x)$. In addition, $z = \bigvee_{x \in \bigcup_{i \in I} A_i} z \wedge x$ by Proposition (4.8). Concluding:

$$f(z) = f(\bigvee_{x \in \bigcup_{i \in I} A_i} z \wedge x)) \bigvee_{x \in \bigcup_{i \in I} A_i} f(z \wedge x) \leq \bigvee_{x \in \bigcup_{i \in I} A_i} f(z) \wedge f(x) \leq \bigvee_{x \in \bigcup_{i \in I} A_i} f(x)$$

and by the arbitrariness of $z \in DM(\bigcup_{i \in I} A_i)$:

$$\bigvee_{z \in DM(\bigcup_{i \in I} A_i)} f(z) \leq \bigvee_{x \in \bigcup_{i \in I} A_i} f(x)$$

($\geq$) Let $x \in \bigcup_{i \in I} A_i$. Since f is monotone and $\bigcup_{i \in I} A_i \subseteq DM(\bigcup_{i \in I} A_i)$, we infer:

$$f(x) \leq \bigwedge_{z \in DM(\bigcup_{i \in I} A_i)} f(z)$$

and by definition of join:

$$\bigvee_{x \in \bigcup_{i \in I} A_i} f(x) \leq \bigwedge_{z \in DM(\bigcup_{i \in I} A_i)} f(z)$$

- $\overline{f}$ is such that $I(\overline{f}) \circ \eta_A = f$: Indeed, for every $a \in A$, we get:

$$(I(\overline{f}) \circ \eta_A)(a) = I(\overline{f})(\eta_A(a)) = \overline{f}(\downarrow a) = \bigvee_{x \in (\downarrow a)} f(x) = f(\bigvee_{x \leq a} x) = f(a)$$

- $\overline{f}$ is unique: Let's consider a map $h : DM(A) \longrightarrow \hat{B}$ in $I_{sup}(\mathbf{cHa})$ such that $h(\eta_A(a)) = f(a)$ for every $a \in A$. Recalling that $X = \bigvee_{x \in X} \downarrow x$ for every $X \subseteq A$, we deduce:

$$h(X) = h(\bigvee_{x\in X} \downarrow x) = \bigvee_{x\in X} h(\downarrow x) = \bigvee_{x\in X} h(\eta_A(x)) = \bigvee_{x\in X} f(x) = \overline{f}(X)$$

$\square$

In a way really similar to what has just been done, we can define the embedding functor

$$\begin{array}{llcl} I_{sup}: & \mathbf{Ba} & \longrightarrow & \mathbf{Sup} \\ & A & \longmapsto & A \\ & (\varphi : A \longrightarrow A') & \longmapsto & (\varphi : A \longrightarrow A') \end{array}$$

and the categories $I_{sup}(\mathbf{Ba})$ and $I_{sup}(\mathbf{cBa})$, and we can prove that the embedding functor

$$\begin{array}{llcl} I: & I_{sup}(\mathbf{cBa}) & \longrightarrow & I_{sup}(\mathbf{Ba}) \\ & \hat{A} & \longmapsto & \hat{A} \\ & (\varphi : \hat{A} \longrightarrow \hat{A}') & \longmapsto & (\varphi : \hat{A} \longrightarrow \hat{A}') \end{array}$$

has a left adjoint DM which coincide with the Dedekind-MacNeille operator on the objects of $I_{sup}(\mathbf{Ba})$.

However, in the case of Boolean algebras there is more to it than just that, as Banaschewski and Bruns pointed out ([BB67]): the requirements that the supremum for every object in **Sup** exists and that the maps in **Sup** respect it are redundant.

Definition 4.19. *The category* $\mathbf{Ba}_\vee$ *is the subcategory of* **Ba** *where maps are homomorphisms of Boolean algebras respecting the existing infinitary joins. The category* $\mathbf{cBa}_\vee$ *is the full subcategory of* $\mathbf{Ba}_\vee$ *where the objects are complete Boolean algebras.*

As before, we can define the embedding functor

$$\begin{array}{llcl} I: & \mathbf{cBa}_\vee & \longrightarrow & \mathbf{Ba}_\vee \\ & \hat{A} & \longmapsto & \hat{A} \\ & (\varphi : \hat{A} \longrightarrow \hat{A}') & \longmapsto & (\varphi : \hat{A} \longrightarrow \hat{A}') \end{array}$$

Theorem 4.6. *Let's consider the categories* $\mathbf{Ba}_\vee$ *and* $\mathbf{cBa}_\vee$*, the embedding functor* I *and the Dedekind-MacNeille operator* DM*. The functor* I *has a left adjoint, which we call* DM*, being the Dedekind-MacNeille operator on the objects of* $\mathbf{Ba}_\vee$*.*

Proof. Let $A \in Ob(\mathbf{Ba}_\vee)$. Then $DM(A) \in Ob(\mathbf{cBa}_\vee)$ by Theorem (4.4). We define the mapping

$$\begin{array}{lccc} \eta_A: & A & \longrightarrow & DM(A) \\ & a & \longmapsto & \downarrow a \end{array}$$

and we need to show that for every $f : A \longrightarrow I(\hat{B}) = \hat{B}$ in $\mathbf{Ba}_\vee$ there exists a unique $\overline{f} : DM(A) \longrightarrow \hat{B}$ in $\mathbf{cBa}_\vee$ such that $I(\overline{f}) \circ \eta_A = f$. We proceed in the same way of the proof of Theorem (4.5): we define $\overline{f} : DM(A) \longrightarrow \hat{B}$ by setting $\overline{f}(X) := \bigvee_{x\in X} f(x)$ for every $X \in DM(A)$, and doing so we get a well-defined monotone unique map such that $\overline{f}(\eta_A(a)) = f(a)$ for every $a \in A$ and such that it respects the existing infinitary joins (all these facts can be shown in the same way as in the proof of Theorem (4.5)). In order to show that $\overline{f}$ is a map in $\mathbf{cBa}_\vee$, it remains to show that $\overline{f}$ is a homomorphism of Boolean algebras (it is no longer trivial in this case). By Lemma (4.6) it suffices to show the following facts:

- $\overline{f}(0_{DM(A)}) = 0_{\hat{B}}$: Indeed, owing to the fact that f is a homomorphism of Boolean algebras, we infer:

$$\overline{f}(0_{DM(A)}) = \overline{f}(\{0\}) = \bigvee_{x\in\{0\}} f(x) = f(0_A) = 0_{\hat{B}}$$

- $\overline{f}(1_{DM(A)}) = 1_{\hat{B}}$: Indeed, as we pointed out in the proof of Theorem (4.5):

$$\overline{f}(1_{DM(A)}) = \overline{f}(A) = \bigvee_{a \in A} f(a) = \bigvee_{\hat{b} \in \hat{B}} \hat{b} = 1_{\hat{B}}$$

- $\overline{f}(X \cup X') = \overline{f}(X) \vee \overline{f}(X')$ for every $X, X' \in Ob(\mathbf{cBa}_{\vee})$. Indeed:

$$\overline{f}(X \cup X') = \bigvee_{z \in X \cup X'} f(z) = \bigvee_{x \in X} f(x) \vee \bigvee_{x' \in X'} f(x') = \overline{f}(X) \vee \overline{f}(X')$$

- $\overline{f}(\neg X) = \neg \overline{f}(X)$ for every $X \in Ob(\mathbf{cBa}_{\vee})$: We need to show that:

$$\overline{f}(X) \wedge \overline{f}(\neg X) = 0_{\hat{B}} \text{ and } \overline{f}(X) \vee \overline{f}(\neg X) = 1_{\hat{B}}$$

  Using the distributive property we get:

$$\overline{f}(X) \wedge \overline{f}(\neg X) = \bigvee_{x \in X} f(x) \wedge \bigvee_{z \in \neg X} f(z) = \bigvee_{x \in X} \bigvee_{z \in \neg X} (f(x) \wedge f(z))$$

  and after recalling that $\neg X = X \to 0 = \{z \in A | \forall x \in X.(z \leq \neg x)\}$ we deduce:

$$\bigvee_{x \in X} \bigvee_{z \in \neg X} (f(x) \wedge f(z)) \leq \bigvee_{x \in X} (f(x) \wedge f(\neg x)) = \bigvee_{x \in X} (f(x \wedge \neg x)) = f(0_A) = 0_{\hat{B}}$$

  since f is a homomorphism of Boolean algebras. In a similar fashion:

$$\overline{f}(X) \vee \overline{f}(\neg X) = \bigvee_{x \in X} \bigvee_{z \in \neg X} (f(x) \vee f(z)) \leq \bigvee_{x \in X} (f(x) \vee f(\neg x)) = \bigvee_{x \in X} (f(x \vee \neg x)) = f(1_A) = 1_{\hat{B}}$$

$\square$

Chapter 5

Preorders

Let's develop an example that is crucial in order to reach our goal. The structure of a preorder turns out to be a really good instance for applying to some concrete structure the categorical notions that have been introduced so far. Besides, preorders are the best place where to introduce some primitive logical elements, such as bottom, top, implication and so on. In fact, the structure of a preorder constitutes the primary shell of a Lindenbaum algebra over a given logical language.

Definition 5.1. *A* ***preordered set*** *(or simply preorder)* $(A, \leq)$ *is a set* A *together with a reflexive transitive binary relation* $\leq$.

A preorder $(A, \leq)$ can be regarded as a category $\mathcal{A}$ whose objects correspond one-to-one to the elements of A and in which for every couple of objects there is at most one map between them, and informally we can think of the unique map $A_1 \longrightarrow A_2$ between the objects A_1 and A_2 in the category as the assertion that $a_1 \leq a_2$ for a_1,a_2 corresponding elements of A. Equivalently, it is possible to define the so called preordered categories as the small categories in which there is at most one map between any two objects. Nevertheless, we are not going to use this definition in this chapter because we prefer to shed a clear light on the parallelism between preorders and correspondent categories, which in fact are preordered categories in the sense just stated. In addition, we can consider the category **Preord** with preorder sets as objects and monotone functions between them as maps.

Definition 5.2. *A* ***partially ordered set*** *(poset)* $(A, \leq)$ *is a preorder with the property that if* $x \leq y$ *and* $y \leq x$ *then* $x = y$*, for every* $x, y \in A$.

As it happens with the preorders, any poset can be regarded as a category, in which the following property holds: for every $X, Y \in Ob(\mathcal{A})$ if $X \cong Y$ then $X = Y$. In addition, we can consider the category **Poset** with objects the partially ordered sets and maps the monotone functions between them.

Lemma 5.1. *Let* A*,* B *be preorders and let* $\mathcal{A}$*,* $\mathcal{B}$ *be the corresponding categories. Then a functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$ *amounts to a monotone function* $f : A \longrightarrow B$*, which is increasing if* F *is covariant, decreasing if* F *is contravariant. Conversely, a monotone function* $f : A \longrightarrow B$ *gives rise to a functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$*, which is covariant if* f *is increasing, contravariant if* f *is decreasing.*

Proof. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a covariant functor. A map $A_1 \longrightarrow A_2$ in $\mathcal{A}$ is sent into a map $F(A_1) \longrightarrow F(A_2)$ in $\mathcal{B}$. We define a function $f : A \longrightarrow B$, $a \longmapsto f(a)$ where $f(a)$ is the element of B corresponding to the object $F(A) \in Ob(\mathcal{B})$, when a is the element of A corresponding to the object $A \in Ob(\mathcal{A})$. Since $A_1 \longrightarrow A_2$ in $\mathcal{A}$ if and only if $a_1 \leq a_2$ in A and $F(A_1) \longrightarrow F(A_2)$ in $\mathcal{B}$ if and only if $f(a_1) \leq f(a_2)$ in B, we infer that $a_1 \leq a_2 \Longrightarrow f(a_1) \leq f(a_2)$. Thus f is an increasing monotone function from A to B.

Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a contravariant functor. A map $A_1 \longrightarrow A_2$ in $\mathcal{A}$ is sent into a map $F(A_2) \longrightarrow F(A_1)$ in $\mathcal{B}$. We define a function $f : A \longrightarrow B$, $a \longmapsto f(a)$ where $f(a)$ is the element of B corresponding to the object $F(A) \in Ob(\mathcal{B})$, when a is the element of A corresponding to the object $A \in Ob(\mathcal{A})$. Since $A_1 \longrightarrow A_2$ in $\mathcal{A}$ if and only if $a_1 \leq a_2$ in A and $F(A_2) \longrightarrow F(A_1)$ in $\mathcal{B}$ if and only if $f(a_2) \leq f(a_1)$ in B, we infer that $a_1 \leq a_2 \Longrightarrow f(a_2) \leq f(a_1)$. Thus f is a decreasing monotone function from A to B.

Let's consider an increasing monotone function $f : A \longrightarrow B$. We define a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ in the following manner. If $A_1 \in Ob(\mathcal{A})$ and a_1 is its corresponding element in the preorder A, then $F(A_1) \in Ob(\mathcal{B})$ is the object of $\mathcal{B}$ corresponding to the element $f(a_1)$ in the preorder B. Let $A_1 \longrightarrow A_2$ be a map in the category $\mathcal{A}$, thus $a_1 \leq a_2$ in the preorder A where a_1, a_2 in A correspond to A_1, A_2 in $Ob(\mathcal{A})$, therefore $f(a_1) \leq f(a_2)$ in B since f is increasing and so we can find a map $F(A_1) \longrightarrow F(A_2)$ in $\mathcal{B}$, where $f(a_1)$, $f(a_2)$ correspond to $F(A_1)$, $F(A_2)$. By the monotonicity of f, it can be easily proved that F respects composition and identities, so it is indeed a covariant functor.

Let's consider a decreasing monotone function $f : A \longrightarrow B$. We define a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ in the following manner. If $A_1 \in Ob(\mathcal{A})$ and a_1 is its corresponding element in the preorder A, then $F(A_1) \in Ob(\mathcal{B})$ is the object of $\mathcal{B}$ corresponding to the element $f(a_1)$ in the preorder B. Let $A_1 \longrightarrow A_2$ be a map in the category $\mathcal{A}$, thus $a_1 \leq a_2$ in the preorder A where a_1, a_2 in A correspond to A_1, A_2 in $Ob(\mathcal{A})$, therefore $f(a_2) \leq f(a_1)$ in B since f is decreasing and so we can find a map $F(A_2) \longrightarrow F(A_1)$ in $\mathcal{B}$, where $f(a_1)$, $f(a_2)$ correspond to $F(A_1)$, $F(A_2)$. By the monotonicity of f, it can be easily proved that F respects composition and identities, so it is indeed a contravariant functor. $\square$

This result makes the definition of the following functor legit:

$$\begin{array}{cccc} I: & \mathbf{Preord} & \longrightarrow & \mathbf{Cat} \\ & (A,\leq) & \mapsto & \mathcal{A} \\ & (A \longrightarrow B) & \mapsto & (\mathcal{A} \longrightarrow \mathcal{B}) \end{array}$$

where $I(A) = \mathcal{A}$ is the preordered categories corresponding to the preorder A. I is a full functor since for every A, B preorders and $f : A \longrightarrow B$ increasing function between them there exists a map $F : \mathcal{A} \longrightarrow \mathcal{B}$ in **Cat**, thanks to Lemma (5.1). Hence **Preord** is a full subcategory of **Cat**. We can also consider the functor:

$$\begin{array}{cccc} R: & \mathbf{Cat} & \longrightarrow & \mathbf{Preord} \\ & \mathcal{A} & \mapsto & (A_{\mathcal{A}},\leq_{\mathcal{A}}) \\ & (\mathcal{A} \longrightarrow \mathcal{B}) & \mapsto & (A_{\mathcal{A}} \longrightarrow B_{\mathcal{B}}) \end{array}$$

where if $\mathcal{A}$ is a small category then $A_{\mathcal{A}}$ is a set containing an element for each object of $\mathcal{A}$ and for a_1,$a_2 \in A$ $a_1 \leq_{\mathcal{A}} a_2$ if and only if there exists a map $A_1 \longrightarrow A_2$ in $\mathcal{A}$ (where A_1, A_2 are the objects corresponding to the elements a_1, a_2). For every map in **Cat**, Lemma (5.1) detects a monotone function between preorders.

Theorem 5.1. *The functor $R : \mathbf{Cat} \longrightarrow \mathbf{Preord}$ is left adjoint to the functor $I : \mathbf{Preord} \longrightarrow \mathbf{Cat}$:*

$$R \dashv I$$

Proof. We define the natural transformation $\eta : 1_{\mathbf{Cat}} \longrightarrow I \circ R$: for every small category $\mathcal{A}$, $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow IR(\mathcal{A})$ sends any object $A \in Ob(\mathcal{A})$ in A and any map $A_1 \longrightarrow A_2$ in itself. It is easy to check that $\eta_{\mathcal{A}}$ is well-defined for every small category $\mathcal{A}$. Let $\mathcal{A}$ be a small category and B a preorder and let $F : \mathcal{A} \longrightarrow I(B)$ be a map in **Cat**, thus a functor between categories $\mathcal{A}$ and $I(B) = \mathcal{B}$. Let $g : R(\mathcal{A}) \longrightarrow B$ be the monotone function corresponding to f, according to Lemma (5.1). Then g is uniquely defined and owing to the definition of I it is such that $I(g) \circ \eta_{\mathcal{A}} = f$. $\square$

Let $(A, \leq)$ be a preorder. We define the relation $\sim$ on A by:

$$\forall a, a' \in A(a \sim a' \Leftrightarrow a \leq a' \text{ and } a' \leq a)$$

It is an equivalence relation, thus we can consider the quotient set $A/\sim$ whose elements are the equivalent classes $[a] = \{a' \in A | a \sim a'\}$. It is straightforward to check that $(A/\sim, \leq)$ is a poset.

Theorem 5.2. *The functor*

$$\begin{array}{rccc} I: & \mathbf{Ord} & \longrightarrow & \mathbf{Preord} \\ & (A,\leq) & \mapsto & (A,\leq) \\ & (A\longrightarrow B) & \mapsfrom & (A\longrightarrow B) \end{array}$$

is left adjoint to the functor

$$\begin{array}{rccc} R: & \mathbf{Preord} & \longrightarrow & \mathbf{Ord} \\ & A & \mapsto & A/\sim \\ & (A\longrightarrow B) & \mapsfrom & (A/\sim\longrightarrow B\sim) \end{array}$$

Proof. We need to show that $I \dashv R$. We define $\eta : 1_{\mathbf{Ord}} \longrightarrow R\circ I$: for every preorder A, $\eta_A : A \longrightarrow RI(A) = A/\sim = A$ is defined by $a \longmapsto [a] = a$. Let A be an order and B a preorder, let $f : A \longrightarrow R(B) = B/\sim$ be a monotone function between orders. Then there is a unique $g : I(A) = A \longrightarrow B$ monotone function between preorders defined by $a \longmapsto f(a)$ and such that $f = R(g)\circ\eta_A$. It is easy to check that g is a well-defined monotone function, moreover $R(g)(\eta_A(a)) = R(g)([a]) = g(a) = f(a)$ for all $a \in A$ and if $h : A \longrightarrow B$ is another such function, then $g(a) = f(a) = R(h)(\eta_A(a)) = R(h)([a]) = h(a)$. □

Composing the two adjunctions of the previous results we can obtain an adjunction between the categories **Cat** and **Ord**. Let's investigate now the notion of natural transformation for preorders.

Lemma 5.2. *Let A, B be preorders and let $\mathcal{A}$, $\mathcal{B}$ be the corresponding categories. Let $f,g : A \longrightarrow B$ be order preserving maps and let $F,G : \mathcal{A} \longrightarrow \mathcal{B}$ be the corresponding functors. Then there is at most one natural transformation* $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{B} \end{tikzcd}$ *and there is exactly one if and only if $f(a) \leq g(a)$ for all $a \in A$.*

Proof. First of all, we notice that the condition (1.1) holds automatically in a preorder, because all diagrams commute. Therefore the natural transformation $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{B} \end{tikzcd}$ is given by $(\alpha_A : F(A) \longrightarrow G(A))_{A\in Ob(\mathcal{A})}$, provided the latter corresponds to an order preserving map, that is $f(a) \leq g(a)$ for all $a \in A$. □

It follows that B^A is a preorder too: its elements are the order-preserving maps from A to B and $f \leq g$ if and only if $f(a) \leq g(a)$ for all $a \in A$. Its corresponding category is the functor category $\mathcal{B}^{\mathcal{A}}$.

Definition 5.3. *Let $(A,\leq)$ be a preorder. An element $t \in A$ is called* ***top*** *of the preorder if $a \leq t$ for all $a \in A$, an element $i \in A$ is called* ***bottom*** *of the preorder if $i \leq a$ for all $a \in A$.*

In a preorder, top and bottom need not be unique, but if the preorder is partially ordered they are unique (if t, t' are bottom of a poset, then $t \leq t'$ and $t' \leq t$, hence $t = t'$).

Lemma 5.3. *Let $(A,\leq)$ be a preorder and $\mathcal{A}$ the correspondent category. Then bottom and top of A correspond respectively to the initial and the terminal object of $\mathcal{A}$.*

Proof. Let $i \in A$ be the bottom of the preorder $(A,\leq)$, so $i \leq a$ for all $a \in A$. If $I \in Ob(\mathcal{A})$ is the object corresponding to i, then there exists exactly one map $I \longrightarrow A$ for all $A \in Ob(\mathcal{A})$. Therefore I is initial in the category $\mathcal{A}$. Similarly for the top of the preorder. □

Whereas top and bottom of a preorders are given by special objects, such as the initial and terminal ones, infimum and supremum are given by some constructions applied to the objects of the category.

Definition 5.4. *Let $(A, \leq)$ be a preorder and let $a, a' \in A$. The* ***meet*** *$a \wedge a'$ of a and a' is an element $\xi \in A$ such that $\xi \leq a$, $z \leq a'$ and for all $z \in A$ such that $z \leq a$, $z \leq a'$ one has that $z \leq \xi$. The* ***join*** *$a \vee a'$ of a and a' is an element $\xi \in A$ such that $a \leq \xi$, $a' \leq \xi$ and for all $z \in A$ such that $a \leq z$, $a' \leq z$ one has that $\xi \leq z$.*

Lemma 5.4. *Let $(A, \leq)$ be a preorder and $\mathcal{A}$ the correspondent category. Let also $a, a' \in A$ and $A, A' \in Ob(\mathcal{A})$ be the corresponding objects. The meet $a \wedge a'$ and the join $a \vee a'$ in the preorder A correspond respectively to the product and the coproduct of A and A' in the category $\mathcal{A}$.*

Proof. Let's consider the product $A \times A'$ of A and A' objects in the category $\mathcal{A}$ and the corresponding element $a \times a'$ in the preorder A. Since there are projections $A \times A' \longrightarrow A$ and $A \times A' \longrightarrow A'$, we have that $a \times a' \leq a$ and $a \times a' \leq a'$. Let $c \in A$ be such that $c \leq a$ and $c \leq a'$, let $C \in Ob(\mathcal{A})$ be the corresponding object in the category $\mathcal{A}$. Then maps $C \longrightarrow A$ and $C \longrightarrow A'$ exist in the category. By definition of binary product, the map $C \longrightarrow A \times A'$ exists. Thus $c \leq a \times a'$ in the preorder A. It follows that $a \times a' = a \wedge a'$.
Conversely, let $a \wedge a' \in A$ and let $A \wedge A' \in Ob(\mathcal{A})$ be the corresponding object. By definition $a \wedge a' \leq a$ and $a \wedge a' \leq a'$, hence maps $A \wedge A' \longrightarrow A$ and $A \wedge A' \longrightarrow A'$ exist. Let's suppose $C \in Ob(\mathcal{A})$ such that $C \longrightarrow A$ and $C \longrightarrow A'$ in the category $\mathcal{A}$. Therefore $c \leq a$ and $c \leq a'$ in the preorder, and so $c \leq a \wedge a'$. As a consequence,the map $C \longrightarrow A \wedge A'$ exists in $\mathcal{A}$, and it is also unique. Besides, the equalities $(A \wedge A' \longrightarrow A) \circ (C \longrightarrow A \wedge A') = (C \longrightarrow A)$ and $(A \wedge A' \longrightarrow A') \circ (C \longrightarrow A \wedge A') = (C \longrightarrow A')$ are clearly proved by correspondence in the preorder A. Finally $A \times A' = A \wedge A'$.

Let's consider the coproduct $A \oplus A'$ of A and A' objects in the category $\mathcal{A}$ and the corresponding element $a \oplus a'$ in the preorder A. Since there are maps $A \longrightarrow A \oplus A'$ and $A' \longrightarrow A \oplus A'$, we have that $a \leq a \oplus a'$ and $a' \leq a \oplus a'$. Let $c \in A$ be such that $a \leq c$ and $a' \leq c$, let $C \in Ob(\mathcal{A})$ be the corresponding object in the category $\mathcal{A}$. Then maps $A \longrightarrow C$ and $A' \longrightarrow C$ exist in the category. By definition of binary coproduct, the map $A \oplus A' \longrightarrow C$ exists. Thus $a \oplus a' \leq c$ in the preorder A. It follows that $a \oplus a' = a \vee a'$.
Conversely, let $a \vee a' \in A$ and let $A \vee A' \in Ob(\mathcal{A})$ be the corresponding object. By definition $a \leq a \vee a'$ and $a' \leq a \vee a'$, hence maps $A \longrightarrow A \vee A'$ and $A' \longrightarrow A \vee A'$ exist. Let's suppose $C \in Ob(\mathcal{A})$ such that $A \longrightarrow C$ and $A' \longrightarrow C$ in the category $\mathcal{A}$. Therefore $a \leq c$ and $a' \leq c$ in the preorder, and so $a \vee a' \leq c$. As a consequence, the map $A \vee A' \longrightarrow C$ exists in $\mathcal{A}$, and it is also unique. Besides, the equalities $(A \vee A' \longrightarrow C) \circ (A \longrightarrow A \vee A') = (A \longrightarrow C)$ and $(A \vee A' \longrightarrow C) \circ (A' \longrightarrow A \vee A') = (A' \longrightarrow C)$ are clearly proved by correspondence in the preorder A. Finally $A \oplus A' = A \vee A'$. □

Proposition 5.1. *Let A, B be preorders and let $\mathcal{A}$, $\mathcal{B}$ be the corresponding categories. An adjunction between $\mathcal{A}$ and $\mathcal{B}$ consists of order-preserving maps $A \underset{g}{\overset{f}{\rightleftarrows}} B$ such that*

$$\forall a \in A, \forall b \in B(f(a) \leq b \Longleftrightarrow a \leq g(b)). \tag{5.1}$$

Proof. Let $(F, G, \overset{-}{\cdot})$ be an adjunction between $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ are functors and $\overset{-}{\cdot} : \mathcal{B}(F(A), B) \overset{\cong}{\longrightarrow} \mathcal{A}(A, G(B))$ in an isomorphism, natural in A and B, for every $A \in Ob(\mathcal{A})$ and $B \in Ob(\mathcal{B})$. Being A and B preorders, the sets $\mathcal{B}(F(A), B)$ and $\mathcal{A}(A, G(B))$ contain both at most one elements. Thus, by Lemma (5.1), there exist two order-preserving maps $f : A \longrightarrow B$ and $g : B \longrightarrow A$ such that $f(a) \leq b \Longleftrightarrow a \leq g(b)$ for all $a \in A$ and $b \in B$. □

Corollary 5.1. *Let A, B be preorders and let $\mathcal{A}$, $\mathcal{B}$ be the corresponding categories. An adjunction between $\mathcal{A}$ and $\mathcal{B}$ consists of order-preserving maps $A \underset{g}{\overset{f}{\rightleftarrows}} B$ such that*

$$\forall a \in A(a \leq g(f(a))) \text{ and } \forall b \in B(f(g(b)) \leq b)$$

Proof. Let $(F, G, \eta, \varepsilon)$ be an adjunction between $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ are functors and $\eta : 1_{\mathcal{A}} \longrightarrow G \circ F$ and $\varepsilon : F \circ G \longrightarrow 1_{\mathcal{B}}$ are the unit and the counit of the adjunction. This means that there exist two order-preserving maps $f : A \longrightarrow B$ and $g : B \longrightarrow A$. Also, the unit of the adjunction is the statement that $a \leq g(f(a))$ for all $a \in A$ and the counit is the statement that $f(g(b)) \leq b$ for all $b \in B$. The triangle identities are equivalent to claim the equality of two maps in an ordered set with the same domain and codomain, and this fact is always true. □

The previuos results give evidence of an all-important link between adjunctions between two preorders categories and Galois connections.

Definition 5.5. *Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. A **monotone Galois connection** between the posets A and B consists of two monotone functions $f : A \longrightarrow B$ and $g : B \longrightarrow A$, such that*

$$\forall a \in A, \forall b \in B (f(a) \leq b \Longleftrightarrow a \leq g(b))$$

Abusing language, if two functors satisfy the condition in the definition above, we say that they are Galois connected, or that they form a Galois connection. However, the condition of the definition is exactly the statement (5.1). We can consequently state the following crucial result about adjunctions between preordered categories.

Theorem 5.3. *Let A, B be posets and $\mathcal{A}$, $\mathcal{B}$ be the corresponding categories. Let also $F : \mathcal{A} \longrightarrow \mathcal{B}$, $G : \mathcal{B} \longrightarrow \mathcal{A}$ be two order-preserving functors. Then $F \dashv G$ if and only if F and G are Galois connected (more precisely, the corresponding order-preserving maps f and g satisfy the condition (5.1), so they give rise to a Galois connection).*

Having said that, the idea that adjunctions can be defined in many different ways also in the case of preorders is now more evident. More precisely, we can list all of them in the following theorem, which is exactly what Theorem (2.1) asserts in the specific case of preordered categories.

Theorem 5.4. *Let $(A, \leq)$, $(B, \leq)$ be preorders and let $\mathcal{A}$, $\mathcal{B}$ be the corresponding categories. Let $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ be two functors, whose corresponding maps are $A \underset{g}{\overset{f}{\rightleftarrows}} B$.*

(i) *$F \dashv G \Leftrightarrow f$, g are monotone functions such that $a \leq g(f(a))$ and $(a \leq g(b) \Longrightarrow f(a) \leq b)$ for all $a \in A$ and $b \in B$;*

(ii) *$F \dashv G \Leftrightarrow g$ is a monotone function and f g are such that $a \leq g(f(a))$ and $(a \leq g(b) \Longrightarrow f(a) \leq b)$ for all $a \in A$ and $b \in B$;*

(iii) *$F \dashv G \Leftrightarrow f$, g are monotone functions such that $f(g(b)) \leq b$ and $(f(a) \leq b \Longrightarrow a \leq g(b))$ for all $a \in A$ and $b \in B$;*

(iv) *$F \dashv G \Leftrightarrow f$ is a monotone function such that $f(g(b)) \leq b$ and $(f(a) \leq b \Longrightarrow a \leq g(b))$ for all $a \in A$ and $b \in B$;*

(v) *$F \dashv G \Leftrightarrow f$, g are monotone functions such that $a \leq g(f(a))$ for all $a \in A$ and $f(g(b)) \leq b$ for all $b \in B$.*

Proof. It follows directly from Theorem (2.1). □

The notion of adjunction becomes crucial since it can be used to characterize top, bottom, the meet and the join of two elements in a preorder, as the following results show.

- *top*
  We define $\mathbf{1}$ to be the category with a unique object $\bullet$ and a unique map $\bullet \longrightarrow \bullet$. Let $(A, \leq)$ be a preorder and let t be its top, let $T \in Ob(\mathcal{A})$ be the corresponding object in the category. Let's consider the functor *top*:

$$\begin{array}{rccc} top: & \mathbf{1} & \longrightarrow & \mathcal{A} \\ & \bullet & \mapsto & T \\ & (\bullet \longrightarrow \bullet) & \mapsto & (1_T : T \longrightarrow T) \end{array}$$

and the trivial functor !:

$$\begin{array}{cccc} !: & \mathcal{A} & \longrightarrow & \mathbf{1} \\ & A & \mapsto & \bullet \\ & (A_1 \longrightarrow A_2) & \mapsto & (\bullet \longrightarrow \bullet) \end{array}$$

Proposition 5.2. *Let $(A, \leq)$ be a preorder with t its top, and let $\mathcal{A}$ be the corresponding category with terminal object T. The trivial functor ! is left adjoint to the functor top (! $\dashv$ top). Conversely, if ! $\dashv$ top then the category $\mathcal{A}$ has T as terminal object (equivalently, the preorder $(A, \leq)$ has t as its top).*

Proof. We need to show that there is a natural isomorphism

$$\mathbf{1}(!(A), \bullet) \cong \mathcal{A}(A, top(\bullet))$$

for every object $A \in Ob(\mathcal{A})$. Being $\mathcal{A}$ the category corresponding to the preorder $(A, \leq)$, it suffices to show that for every $a \in A$:

$$!(a) \leq \bullet \Leftrightarrow a \leq top(\bullet)$$

which is equivalent to ask that $a \leq t$ for every $a \in A$, which for sure holds by definition of t top in the preorder $(A, \leq)$. Reversing the same reasoning, one gets the second part of the Proposition. □

As a consequence of the previous Proposition, in a category $\mathcal{C}$ the terminal object can be described by an adjunction.

- *bottom*
  We define $\mathbf{1}$ to be the category with a unique object $\bullet$ and a unique map $\bullet \longrightarrow \bullet$. Let $(A, \leq)$ be a preorder and let i be its bottom, let $I \in Ob(\mathcal{A})$ be the corresponding object in the category. Let's consider the functor *bottom*:

$$\begin{array}{cccc} bottom: & \mathbf{1} & \longrightarrow & \mathcal{A} \\ & \bullet & \mapsto & I \\ & (\bullet \longrightarrow \bullet) & \mapsto & (1_I : I \longrightarrow I) \end{array}$$

and the trivial functor !.

Proposition 5.3. *Let $(A, \leq)$ be a preorder with i its bottom, and let $\mathcal{A}$ be the corresponding category with initial object I. The functor bottom is left adjoint to the trivial functor ! (bottom $\dashv$!). Conversely, if bottom $\dashv$! then the category $\mathcal{A}$ has I as initial object (equivalently, the preorder $(A, \leq)$ has i as its bottom).*

Proof. We need to show that there is a natural isomorphism

$$\mathcal{A}(bottom(\bullet), A) \cong \mathbf{1}(\bullet, !(A))$$

for every object $A \in Ob(\mathcal{A})$. Being $\mathcal{A}$ the category corresponding to the preorder $(A, \leq)$, it suffices to show that for every $a \in A$:

$$bottom(\bullet) \leq a \Leftrightarrow \bullet \leq !(a)$$

which is equivalent to ask that $i \leq a$ for every $a \in A$, which for sure holds by definition of i bottom in the preorder $(A, \leq)$. Reversing the same reasoning, one gets the second part of the Proposition. □

As a consequence of the previous Proposition, in a category $\mathcal{C}$ the initial object can be described by an adjunction.

- *meet*
Let $(A, \leq)$ be a preorder where the meet of any two elements exists. The preorder $(A \times A, \leq)$ has elements $\{(a, \widetilde{a}) | a, \widetilde{a} \in A\}$ and $(a, \widetilde{a}) \leq (a', \widetilde{a}') \Leftrightarrow (a \leq a'$ and $\widetilde{a} \leq \widetilde{a}')$. Let $\mathcal{A}$ be the category corresponding to the preorder A and $\mathcal{A} \times \mathcal{A}$ the category corresponding to the preorder $A \times A$. The diagonal functor *diag* is defined as:

$$\begin{array}{cccc} diag: & \mathcal{A} & \longrightarrow & \mathcal{A} \times \mathcal{A} \\ & A & \mapsto & (A, A) \\ & (A_1 \longrightarrow A_2) & \mapsto & ((A_1, A_1) \longrightarrow (A_2, A_2)) \end{array}$$

and the product functor $\times$ is defined as:

$$\begin{array}{cccc} \times: & \mathcal{A} \times \mathcal{A} & \longrightarrow & \mathcal{A} \\ & (A, \widetilde{A}) & \mapsto & A \times \widetilde{A} \\ & ((A, \widetilde{A}) \longrightarrow (A', \widetilde{A}')) & \mapsto & (A \times \widetilde{A} \longrightarrow A \times \widetilde{A}')) \end{array}$$

Proposition 5.4. *Let $(A, \leq)$ be a preorder where the meet of any two elements exists, and let $\mathcal{A}$ be the corresponding category (where consequently binary products exist). The functor diag is left adjoint to the functor $\times$ ($diag \dashv \times$). Conversely, if $diag \dashv \times$ then the category $\mathcal{A}$ has binary products (equivalently, the preorder $(A, \leq)$ has meet for every couple of elements).*

Proof. We need to show that there is a natural isomorphism

$$(\mathcal{A} \times \mathcal{A})(diag(A'), (A, \widetilde{A})) \cong \mathcal{A}(A', A \times \widetilde{A})$$

for every $A', A, \widetilde{A} \in Ob(\mathcal{A})$. Since the category $\mathcal{A}$ corresponds to the preorder $(A, \leq)$ and products in the category correspond to meets in the preorder, it suffices to show that for every $a', a, \widetilde{a} \in A$:

$$(a', a') \leq (a, \widetilde{a}) \Leftrightarrow a' \leq a \wedge \widetilde{a}$$

which is true since:

$$(a', a') \leq (a, \widetilde{a}) \Leftrightarrow (a' \leq a \text{ and } a' \leq \widetilde{a}) \Leftrightarrow a' \leq a \wedge \widetilde{a}$$

for every $a', a, \widetilde{a} \in A$. Conversely, let's suppose that $diag \dashv \times$. It follows that in the language of the preorders $(A, \leq)$ and $(A \times A, \leq)$: $(a', a') \leq (a, \widetilde{a}) \Leftrightarrow a' \leq a \wedge \widetilde{a}$ and $a \wedge \widetilde{a} \leq a \wedge \widetilde{a} \Leftrightarrow (a \wedge \widetilde{a}, a \wedge \widetilde{a}) \leq (a, \widetilde{a})$. This means that for every $a', a, \widetilde{a} \in A$: if $a' \leq a$ and $a' \leq \widetilde{a}$ then $a' \leq a \wedge \widetilde{a}$, also $a \wedge \widetilde{a} \leq a$ and $a \wedge \widetilde{a} \leq \widetilde{a}$. □

As a consequence of the previous Proposition, in a category $\mathcal{C}$ the product can be described by an adjunction.

- *join*
Let $(A, \leq)$ be a preorder where the join of any two elements exists. Let $\mathcal{A}$ be the category corresponding to the preorder A. The coproduct functor $\oplus$ is defined as:

$$\begin{array}{cccc} \oplus: & \mathcal{A} \times \mathcal{A} & \longrightarrow & \mathcal{A} \\ & (A, \widetilde{A}) & \mapsto & A \oplus \widetilde{A} \\ & ((A, \widetilde{A}) \longrightarrow (A', \widetilde{A}')) & \mapsto & (A \oplus \widetilde{A} \longrightarrow A \oplus \widetilde{A}')) \end{array}$$

Proposition 5.5. *Let $(A, \leq)$ be a preorder where the join of any two elements exists, and let $\mathcal{A}$ be the corresponding category (where consequently binary coproducts exist). The functor $\oplus$ is left adjoint to the functor diag ($\oplus \dashv diag$). Conversely, if $\oplus \dashv diag$ then the category $\mathcal{A}$ has binary coproducts (equivalently, the preorder $(A, \leq)$ has join for every couple of elements).*

Proof. We need to show that there is a natural isomorphism

$$\mathcal{A}(A \oplus \widetilde{A}, A') \cong (\mathcal{A} \times \mathcal{A})((A, \widetilde{A}), diag(A'))$$

for every $A', A, \widetilde{A} \in Ob(\mathcal{A})$. Since the category $\mathcal{A}$ corresponds to the preorder $(A, \leq)$ and coproducts in the category correspond to joins in the preorder, it suffices to show that for every $a', a, \widetilde{a} \in A$:

$$a \vee \widetilde{a} \leq a' \Leftrightarrow (a, \widetilde{a}) \leq (a', a')$$

which is true since:

$$(a, \widetilde{a}) \leq (a', a') \Leftrightarrow (a \leq a' \text{ and } \widetilde{a} \leq a') \Leftrightarrow a \vee \widetilde{a} \leq a'$$

for every $a', a, \widetilde{a} \in A$. Conversely, let's suppose that $\oplus \dashv diag$. It follows that in the language of the preorders $(A, \leq)$ and $(A \times A, \leq)$: $(a, \widetilde{a}) \leq (a', a') \Leftrightarrow a \vee \widetilde{a} \leq a'$ and $a \vee \widetilde{a} \leq a \vee \widetilde{a} \Leftrightarrow (a, \widetilde{a}) \leq (a \vee \widetilde{a}, a \vee \widetilde{a})$. This means that for every $a', a, \widetilde{a} \in A$: if $a \leq a'$ and $\widetilde{a} \leq a'$ then $a \vee \widetilde{a} \leq a'$, also $a \leq a \vee \widetilde{a}$ and $\widetilde{a} \leq a \vee \widetilde{a}$. □

As a consequence of the previous Proposition, in a category $\mathcal{C}$ the coproduct can be described by an adjunction.

So far, we have shown that categorical language is powerful enough to deal with preorders and posets and thus to describe categorically their structures. Essentially, this does suggest that Heyting and Boolean algebras, hence (only propositional, for now) intuitionistic and classical logic, can be treated within the setting of category theory, as we will see at the very end of the next chapter.

Chapter 6

Logical connectives as adjoint functors

The main goal of this chapter is to use the categorical language and the tools of category theory for the treatment of propositional logic. In order to do so, we need to give a categorical characterization (i.e. by means of adjoint functors) of the intuitionistic and classical connectives. Once this is worked out, it will be possible to give the categorical definitions of Heyting and Boolean algebras, as particular cases of ordered categories. This is the very starting point of categorical logic, whose origins will be presented and whose developments will be sketched.

6.1 The adjoint-functorial nature of connectives: categorical logic

As already mentioned, the turning point towards a categorical treatment of logic, which in fact is categorical logic, consists in making explicit the hidden essentially adjoint-functorial nature of logical connectives and quantifiers. We recall that category theory was invented in 1945 by Eilenberg (September 30, 1913 – January 30, 1998) and Saunders Mac Lane (4 August 1909 – 14 April 2005), making its first official appearance in the paper [ME45]. Immediately it was seen by other mathematicians as a powerful and useful tool to deal with algebraic aspects of mathematics, and by the end of the 1950s its concepts and methods had become the full-fledged canonical way to treat algebraic topology and algebraic geometry.

6.1.1 The birth of categorical logic

J. Lambek

At the beginning, logic and category theory had nothing in common, and logicians of those days were most of the times totally unacquainted with category theory. The first time these two subjects had a contact was in the mind of an algebraist, Joachim "Jim" Lambek (5 December 1922 – 23 June 2014). He first discerned analogies between the way a category is defined and deductive systems in the style of those introduced by Gerhard Karl Erich Gentzen (November 24, 1909 – August 4, 1945) in 1930s. This hint was taken in earnest by Francis William Lawvere (February 9, 1937), who is rightly regarded as the founder of categorical logic: in his pioneering work, which came to be known as **functorial semantics** ([Law63]), he thoroughly reshape logical semantics and syntax in categorical terms, identifying algebraic theories as categories with certain properties, interpretations as functors between them and models as functors from them to the category **Set** of

sets. Lawvere's project was however far more complex. He actually began to extend this description of algebraic theories in category-theoretic terms to first-order theories (also called elementary theories), after introducing quantification as a particular kind of adjunction. In fact, we was a staunch supporter of the idea that category theory could serve as a foundation for mathematics: on the one hand he had just discover a possible way for the development of logic through category-theoretic language and tools, on the other he was convinced that category theory reflected the real essence of mathematical (and not only algebraic) structures. Together with Myles Tierney (1937), he started to look into the consequences of taking as an axiom the existence of an object of truth value, which was necessary in order to express quantification. Together they ended up with the foundation of a concept of terrific fertility: the elementary topos, i.e. a cartesian closed category with an object of truth value ([Law70b], [Tie72], [Tie73]). Lawvere and Tierney's ideas were taken up and further developed by several mathematicians and logicians, and the belief that logic can be done into a topos turned out soon to be well-founded: a topos' internal logic is a form of intuitionistic type theory and conversely each type theory generates a topos.
We can say that these two crucial steps, that is the foundation of functorial semantics by Lawvere and the formulation of the concept of elementary topos by Lawvere and Tierney, officially establish the birth of categorical logic, as we mean it today.

It is worth underlining that categorical logic did not develop as a stand-alone and self-supporting branch of logic (or mathematics). Rather, as researching and studying of this subject went along, connections and stimuli to many other fields of mathematics, theoretical computer science and logic came to light more and more often. Among the others, fields such as intuitionistic Zermelo-Fraenkel set theory (IZF), philosophy of mathematics, recursion theory, formal topology, type theory, and algebraic analysis of deductive systems, were deeply and profitably influenced by the outcomes of categorical logic.

6.1.2 Lawvere's pioneering work: the "categorification" of logic

As we have already said, it is unquestionable that we can ascribe the merit of the foundation of categorical logic to F. William Lawvere, and we can settle its birth in 1963 with the defence of his PhD Thesis under the Eilenberg's supervision ([Law63]).

All the ideas through which Lawvere has influenced the categorical community can be found in this paper. In a nutshell they are:

1. the category of categories **CAT** should be the foundations of mathematics;

2. every aspect of mathematics should be expressed in category-theoretic terms;

3. mathematical constructions should be thought of as functors;

4. there is no such thing as a mathematical concept by itself (in particular there are no such thing as sets by themselves, since they always appear in a category);

5. the category **Set** should play a key role in the category **CAT** of categories;

6. adjunction plays a pivotal role in the development of mathematics (one possible and fruitful way of doing research is looking for adjoints to given functors);

7. the invariant character of any mathematical procedure or construction should be expressed in terms of functoriality;

8. the real objective content of a mathematical theory is its invariant content, so functoriality is the language through which a mathematical theory should be qualified;

9. logical aspects of the foundations of mathematics in particular should be treated categorically, and thus they should be revealed by adjoint functors.

F. W. Lawvere

The above list shows clearly a change of the status of the notion of category among category theorists in 1960s and 1970s. In Lawvere's visions, categories constituted "autonomous types and [were], as such, independent of any underlying set-theoretical structures and structure preserving functions" ([MR11], pg 11). In addition to that, Lawvere started looking at categories also as algebraic descriptions of formal systems, and so as formal systems themselves. The first striking new idea of Lawvere came when he realized that certain categories could be defined by asserting the existence of certain adjoint functors to given elementary functors, therefore by asserting that certain basic conceptual operations are representable in those categories. The point was that the so-defined categories turned out to retrace logical concepts and theories: the existence of specific adjoint functors to appropriate given functors was equivalent to a specification of particular kinds of logical structures. The step that separated these facts from the program of "functorizing" the study of mathematical, and above all logical, concepts appeared immediately in front of Lawvere's eyes.

Although 1963 Lawvere's thesis contained the seeds and ideas of what would be better formalized in an out-and-out program for categorical logic a couple of years later, the concepts developed in it are only a small part of those listed above. Actually, Lawvere treated in his PhD thesis only **algebraic categories**, **algebraic theories**[1] and **algebraic functors**. The main tool he used and the key to connect these three notions together are adjoint functors. Thanks to them, he proved that every algebraic functor[2] has a left adjoint, that algebraic semantics[3] has a left adjoint, which can be called **algebraic structure**, and he succeeded in giving a categorical characterization of algebraic categories.

A summary of Lawvere's thesis was communicated to the *Proceedings of the National Academy of Science* by MacLane already in 1963. The following year MacLane communicated to the same *Proceedings* also Lawvere's attempt at the axiomatization of the Elementary Theory of the Category of Sets (ECTS), which had not appeared in the 1963 thesis. The role of ETCS was of crucial importance: if on the one hand it gave suggestions to Lawvere and others about how logic could be developed in a categorical setting, on the other it appeared as a counterpart of elementary topos theory (ETT) for the common purpose of developing a categorical description of sets and it highlighted many strengths of ETT that otherwise could have been underrated (among them, the fact that ETT bridged the gap between a categorical description of sets and the method of forcing). Indeed, at the time ETCS was viewed as a simple translation in category-theoretic terms of the axioms of set theory and so it seemed that it could not provide any new insight into set theory itself. Also, it is worth mentioning that sociological and historical factors should probably have played a role: in those years Cohen's proof on the independence of continuum hypothesis was published and the brand new method of forcing attracted (deservedly) the interests of the bulk of set theorists.

Be that as it may, it is important to recognize that the very first hint to develop such an ambitious project, i.e. categorical logic, came to Lawvere by noticing exactly what we showed in the previous chapter: posets can be presented as particular categories and maximum, minimum, meets and joins can be introduced by means of adjoints to canonical given functors, such as the trivial and the diagonal functors. That given, the step towards a category-theoretic treatment of

[1] An **algebraic theory** $\mathcal{A}$ is a small category, in which the objects of $\mathcal{A}$ form a countable set $\{A^0, A^1, ..., A^n, ...\}$ of distinct elements such that for all n A^n is the product of A^1 with itself n times (hence the projection maps $\pi_i^{(n)}$ exist for $i = 1, ..., n-1$ and for every n). Also, for each map $\theta_i : A^m \longrightarrow A^1$ $(i = 1, ..., n-1)$ in $\mathcal{A}$ there exists one and only one map $\langle\theta_0, ..., \theta_{n-1}\rangle : A^m \longrightarrow A^n$ such that $\pi_i^{(n)} \times \langle\theta_0, ..., \theta_{n-1}\rangle = \theta_i$ $(i = 1, ..., n-1)$. The functor category $\mathbf{Set}^{\mathcal{A}}$ of all product preserving functors $\mathcal{A} \longrightarrow \mathbf{Set}$ can be thought of as the category of models of the theory $\mathcal{A}$ and it is called an **algebraic category**.

[2] Given a morphism of algebraic theories $f : \mathcal{A} \longrightarrow \mathcal{B}$, the corresponding **algebraic functor** is $\mathbf{Set}^f : \mathbf{Set}^{\mathcal{A}} \longrightarrow \mathbf{Set}^{\mathcal{B}}$.

[3] **Algebraic semantics** is a construction assigning to each algebraic theory $\mathcal{A}$ the forgetful functor $U_{\mathcal{A}}$, and to each map $f : \mathcal{A} \longrightarrow \mathcal{B}$ of algebraic theories the corresponding algebraic functor $\mathbf{Set}^f : \mathbf{Set}^{\mathcal{A}} \longrightarrow \mathbf{Set}^{\mathcal{B}}$.

connectives is really short: posets are the structures at the core of Lindenbaum algebras and the canonical operations in a poset are the bones of logical connectives in Heyting or Boolean algebras. The next section gives a formal setting to these intuitions.

6.2 Logical connectives via adjunctions

In the previuos chapter we looked into the properties of preordered categories. In particular we proved that the notion of adjunction is powerful enough to characterize terminal objects, initial objects, products and coproducts of categories, which correspond respectively to top, bottom, meets and joins of elements in the preoder associated to a preordered category. This is summed up in the following theorem. First of all we recall some all-important notations and definitions.

We define $\mathbf{1}$ to be the category with a unique object $\bullet$ and a unique map $\bullet \longrightarrow \bullet$. If $\mathcal{C}$ is a preordered category with initial object $I \in Ob(\mathcal{C})$ and terminal object $T \in Ob(\mathcal{C})$, we define the functors *top* and *bottom*:

$$\begin{array}{rccc} bottom: & \mathbf{1} & \longrightarrow & \mathcal{C} \\ & \bullet & \mapsto & I \\ & (\bullet \longrightarrow \bullet) & \mapsto & (1_I : I \longrightarrow I) \end{array} \qquad \begin{array}{rccc} top: & \mathbf{1} & \longrightarrow & \mathcal{C} \\ & \bullet & \mapsto & T \\ & (\bullet \longrightarrow \bullet) & \mapsto & (1_T : T \longrightarrow T) \end{array}$$

For any category $\mathcal{C}$ we can also define the trivial functor !:

$$\begin{array}{rccc} ! : & \mathcal{C} & \longrightarrow & \mathbf{1} \\ & C & \mapsto & \bullet \\ & (C_1 \longrightarrow C_2) & \mapsto & (\bullet \longrightarrow \bullet) \end{array}$$

the diagonal functor Δ:

$$\begin{array}{rccc} \Delta : & \mathcal{C} & \longrightarrow & \mathcal{C} \times \mathcal{C} \\ & C & \mapsto & (C, C) \\ & (C_1 \longrightarrow C_2) & \mapsto & ((C_1, C_1) \longrightarrow (C_2, C_2)) \end{array}$$

If the category $\mathcal{C}$ admits products and coproducts, we can finally define the product and coproduct functors $\times$ and $\oplus$:

$$\begin{array}{rccc} \times : & \mathcal{C} \times \mathcal{C} & \longrightarrow & \mathcal{C} \\ & (C, \widetilde{C}) & \mapsto & C \times \widetilde{C} \\ & ((C, \widetilde{C}) \longrightarrow (C', \widetilde{C}')) & \mapsto & (C \times \widetilde{C} \longrightarrow C \times \widetilde{C}') \end{array} \qquad \begin{array}{rccc} \oplus : & \mathcal{C} \times \mathcal{C} & \longrightarrow & \mathcal{C} \\ & (C, \widetilde{C}) & \mapsto & C \oplus \widetilde{C} \\ & ((C, \widetilde{C}) \longrightarrow (C', \widetilde{C}')) & \mapsto & (C \oplus \widetilde{C} \longrightarrow C \oplus \widetilde{C}') \end{array}$$

Theorem 6.1. *Let $\mathcal{C}$ be a preordered category. Then:*

- *$\mathcal{C}$ has terminal object $T \Leftrightarrow\ ! \dashv top$;*
- *$\mathcal{C}$ has initial object $I \Leftrightarrow bottom \dashv !$;*
- *$\mathcal{C}$ admits meets $\Leftrightarrow \Delta \dashv \times$;*
- *$\mathcal{C}$ admits joins $\Leftrightarrow \oplus \dashv \Delta$;*

Our purpose now is showing that the notion of adjunction permits furthermore to speak about logical connectives for intuitionistic and classical propositional logic. More precisely, we will take up the hints given by a category-theoretic treatment of preorders in order to follow in Lawvere's first footsteps and so to begin a categorical treatment of logic by the presentation of connectives as adjoint functors.

Definition 6.1. *Let $\mathcal{L}$ be a language for the intuitionistic propositional logic IL and let $\mathfrak{A}_i(\mathcal{L})$ be the ordered category corresponding to the poset $(\mathfrak{A}_i(\mathcal{L}), \leq)$. We define the **false functor** $\bot$ and the **true functor** $\top$:*

$$\begin{array}{rccc} \bot & \mathbf{1} & \longrightarrow & \mathfrak{A}_i(\mathcal{L}) \\ & \bullet & \mapsto & [\bot] \\ & (\bullet \longrightarrow \bullet) & \mapsto & (1_{[\bot]} : [\bot] \longrightarrow [\bot]) \end{array} \qquad \begin{array}{rccc} \top : & \mathbf{1} & \longrightarrow & \mathfrak{A}_i(\mathcal{L}) \\ & \bullet & \mapsto & [\top] \\ & (\bullet \longrightarrow \bullet) & \mapsto & (1_{[\top]} : [\top] \longrightarrow [\top]) \end{array}$$

*the **conjunction functor** $(-)\wedge(-)$:*

$$\begin{array}{rccc} (-)\wedge(-): & \mathfrak{A}_i(\mathcal{L})\times\mathfrak{A}_i(\mathcal{L}) & \longrightarrow & \mathfrak{A}_i(\mathcal{L}) \\ & ([\varphi],[\psi]) & \mapsto & [\varphi\wedge\psi] \\ & (([\varphi],[\psi])\longrightarrow([\varphi'],[\psi'])) & \mapsto & ([\varphi\wedge\psi]\longrightarrow[\varphi'\wedge\psi']) \end{array}$$

*the **disjunction functor** $(-)\vee(-)$:*

$$\begin{array}{rccc} (-)\vee(-): & \mathfrak{A}_i(\mathcal{L})\times\mathfrak{A}_i(\mathcal{L}) & \longrightarrow & \mathfrak{A}_i(\mathcal{L}) \\ & ([\varphi],[\psi]) & \mapsto & [\varphi\vee\psi] \\ & (([\varphi],[\psi])\longrightarrow([\varphi'],[\psi'])) & \mapsto & ([\varphi\vee\psi]\longrightarrow[\varphi'\vee\psi']) \end{array}$$

*the **implication functor** $(-)^{(-)}$:*

$$\begin{array}{rccc} (-)^{(-)}: & \mathfrak{A}_i(\mathcal{L})\times\mathfrak{A}_i(\mathcal{L}) & \longrightarrow & \mathfrak{A}_i(\mathcal{L}) \\ & ([\varphi],[\psi]) & \mapsto & [\varphi\to\psi] \\ & (([\varphi],[\psi])\longrightarrow([\varphi'],[\psi'])) & \mapsto & ([\varphi\to\psi]\longrightarrow[\varphi'\to\psi']) \end{array}$$

*and the **negation functor** $\neg(-)$:*

$$\begin{array}{rccc} \neg(-): & \mathfrak{A}_i(\mathcal{L}) & \longrightarrow & \mathfrak{A}_i(\mathcal{L})^{op} \\ & [\varphi] & \mapsto & [\neg\varphi] \\ & ([\varphi]\longrightarrow[\varphi']) & \mapsto & ([\neg\varphi']\longrightarrow[\neg\varphi]) \end{array}$$

As we defined the Lindenbaum algebra over a language $\mathcal{L}$ for the intuitionistic predicative logic, we noticed that the category $\mathfrak{A}_i(\mathcal{L})$ is an ordered category, corresponding to the poset $(\mathfrak{A}_i(\mathcal{L}),\leq)$ where:

$$[\varphi]\leq[\psi]\Leftrightarrow\varphi\vdash\psi$$

Consequently, from now on we will denote a morphism $[\varphi]\longrightarrow[\psi]$ in $\mathfrak{A}_i(\mathcal{L})$ by: $[\varphi]\leq[\psi]$. This notation allows us to differentiate clearly the functorial arrow $\longrightarrow$ between categories and the map $\leq$ between objects.

Theorem 6.2. *Let $\mathcal{L}$ be a language for the intuitionistic propositional logic IL and let $\mathfrak{pr}$ be a fixed proposition. Then:*

1. *the true functor $\top$ is right adjoint to the trivial functor $!$:*

$$!\dashv\top$$

2. *the false functor $\bot$ is left adjoint to the trivial functor $!$:*

$$\bot\dashv!$$

3. *the conjunction functor $(-)\wedge(-)$ is right adjoint to the diagonal functor Δ:*

$$\Delta\dashv(-)\wedge(-)$$

4. *the disjunction functor $(-)\vee(-)$ is left adjoint to the diagonal functor Δ:*

$$(-)\vee(-)\dashv\Delta$$

5. *the implication of $\mathfrak{pr}$ functor $(-)^{\mathfrak{pr}}$ is right adjoint to the conjunction of $\mathfrak{pr}$ functor $(-)\wedge\mathfrak{pr}$:*

$$(-)\wedge\mathfrak{pr}\dashv(-)^{\mathfrak{pr}}$$

6. *the negation functor $\neg(-)$ is left adjoint to the opposite negation functor $\neg^{op}(-)$:*

$$\neg(-)\dashv\neg^{op}(-)$$

Proof. 1. By Proposition 5.2, we know that if $[\top]$ is the terminal object of the ordered category $\mathfrak{A}_i(\mathcal{L})$ then the trivial functor is left adjoint to the true functor $\top$. Hence, it suffices to show that $[\varphi] \leq [\top]$ for every $[\varphi] \in Ob(\mathfrak{A}_i(\mathcal{L}))$, which is equivalent to show that $\varphi \vdash \top$ for every formula $\varphi \in Form_{\mathcal{L}}$, and this always holds.

2. By Proposition 5.3, we know that if $[\bot]$ is the initial object of the ordered category $\mathfrak{A}_i(\mathcal{L})$ then the trivial functor is right adjoint to the false functor $\bot$. Hence, it suffices to show that $[\bot] \leq [\varphi]$ for every $[\varphi] \in Ob(\mathfrak{A}_i(\mathcal{L}))$, which is equivalent to show that $\bot \vdash \varphi$ for every formula $\varphi \in Form_{\mathcal{L}}$, and this always holds.

3. Let $([\varphi],[\psi]) \in Ob(\mathfrak{A}_i(\mathcal{L}) \times \mathfrak{A}_i(\mathcal{L}))$. There is map $\Delta(([\varphi]) \wedge ([\psi])) = ([\varphi \wedge \psi],[\varphi \wedge \psi]) \leq ([\varphi],[\psi])$ in $\mathfrak{A}_i(\mathcal{L}) \times \mathfrak{A}_i(\mathcal{L})$, which means a couple of maps $[\varphi \wedge \psi] \leq [\varphi]$ and $[\varphi \wedge \psi] \leq [\psi]$ in $\mathfrak{A}_i(\mathcal{L})$. Indeed:

$$\frac{\overline{\varphi, \psi \vdash \varphi}}{\varphi \wedge \psi \vdash \varphi}\ \wedge l \quad \text{and} \quad \frac{\overline{\varphi, \psi \vdash \psi}}{\varphi \wedge \psi \vdash \psi}\ \wedge l$$

Let $[\alpha] \in Ob(\mathfrak{A}_i(\mathcal{L}))$ and let's suppose there is a map $\Delta([\alpha]) = ([\alpha],[\alpha]) \leq ([\varphi],[\psi])$ in $\mathfrak{A}_i(\mathcal{L}) \times \mathfrak{A}_i(\mathcal{L})$, therefore there are maps $[\alpha] \leq [\varphi]$ and $[\alpha] \leq [\psi]$ in $\mathfrak{A}_i(\mathcal{L})$. We need to find a map $[\alpha] \leq [\varphi \wedge \psi]$ in $\mathfrak{A}_i(\mathcal{L})$, then we need to show that $\alpha \vdash \varphi \wedge \psi$:

$$\frac{\overline{\alpha \vdash \varphi} \qquad \overline{\alpha \dashv \psi}}{\alpha \vdash \varphi \wedge \psi}\ \wedge r$$

where $\alpha \vdash \varphi$ and $\alpha \dashv \psi$ are assured by our assumptions.

4. Let $([\varphi],[\psi]) \in Ob(\mathfrak{A}_i(\mathcal{L}) \times \mathfrak{A}_i(\mathcal{L}))$. There is a map $([\varphi],[\psi]) \leq \Delta([\varphi \vee \psi]) = ([\varphi \vee \psi],[\varphi \vee \psi])$ in $\mathfrak{A}_i(\mathcal{L}) \times \mathfrak{A}_i(\mathcal{L})$, which means a couple of maps $[\varphi] \leq [\varphi \vee \psi]$ and $[\psi] \leq [\varphi \vee \psi]$ in $\mathfrak{A}_i(\mathcal{L})$. Indeed:

$$\frac{\overline{\varphi \vdash \varphi}}{\varphi \vdash \varphi \vee \psi}\ \vee r \quad \text{and} \quad \frac{\overline{\psi \vdash \psi}}{\psi \vdash \varphi \vee \psi}\ \vee r$$

Let $[\alpha] \in Ob(\mathfrak{A}_i(\mathcal{L}))$ and let's suppose there is a map $([\varphi],[\psi]) \leq \Delta([\alpha]) = ([\alpha],[\alpha])$ in $\mathfrak{A}_i(\mathcal{L}) \times \mathfrak{A}_i(\mathcal{L})$, therefore there are maps $[\varphi] \leq [\alpha]$ and $[\psi] \leq [\alpha]$ in $\mathfrak{A}_i(\mathcal{L})$. We need to find a map $[\varphi \vee \psi] \leq [\alpha]$ in $\mathfrak{A}_i(\mathcal{L})$, then we need to show that $\varphi \vee \psi \vdash \alpha$:

$$\frac{\overline{\varphi, \psi \vdash \alpha}}{\varphi \vee \psi \vdash \alpha}\ \vee l$$

where $\varphi, \psi \vdash \alpha$ follows from the existence of maps $[\varphi] \leq [\alpha]$ and $[\psi] \leq [\alpha]$ in $\mathfrak{A}_i(\mathcal{L})$.

5. Let $[\varphi] \in Ob(\mathfrak{A}_i(\mathcal{L}))$. There is a map $[\varphi] \leq (([\varphi]) \wedge (\mathfrak{pr}))^{\mathfrak{pr}} = [\mathfrak{pr} \to (\varphi \wedge \mathfrak{pr})]$ in $\mathfrak{A}_i(\mathcal{L})$, indeed:

$$\frac{\dfrac{\overline{\varphi, \mathfrak{pr} \vdash \varphi} \qquad \overline{\varphi, \mathfrak{pr} \vdash \mathfrak{pr}}}{\varphi, \mathfrak{pr} \vdash \varphi \wedge \mathfrak{pr}}\ \wedge r}{\varphi \vdash \mathfrak{pr} \to (\varphi \wedge \mathfrak{pr})}\ \to r$$

Let $[\alpha] \in Ob(\mathfrak{A}_i(\mathcal{L}))$ and let $[\varphi] \leq [\mathfrak{pr} \to \alpha]$ be a map in $\mathfrak{A}_i(\mathcal{L})$. We need to find a map $[\varphi \wedge \mathfrak{pr}] \leq [\alpha]$, thus it suffices to show that $\varphi \wedge \mathfrak{pr} \vdash \alpha$:

$$\frac{\dfrac{\overline{\varphi \vdash \mathfrak{pr} \to \alpha}}{\varphi, \mathfrak{pr} \vdash \alpha}\ (\to r)^{-1}}{\varphi \wedge \mathfrak{pr} \vdash \alpha}\ \wedge l$$

where $\varphi \vdash \mathfrak{pr} \to \alpha$ is assured by assumption.

6. Let $[\varphi] \in Ob(\mathfrak{A}_i(\mathcal{L}))$. There is a map $[\varphi] \leq \neg^{op}(\neg([\varphi])) = [\neg\neg\varphi]$ in $\mathfrak{A}_i(\mathcal{L})$. Indeed:

$$\dfrac{\dfrac{\overline{\varphi \vdash \varphi} \qquad \overline{\varphi, \bot \vdash \bot}}{\varphi, \varphi \to \bot \vdash \bot} \to l}{\varphi \vdash \neg\neg\varphi = ((\varphi \to \bot) \to \bot)} \to r$$

Let $[\alpha] \in Ob(\mathfrak{A}_i^{op}(\mathcal{L})$ and let $[\varphi] \leq [\neg\alpha]$ be a map in $\mathfrak{A}_i(\mathcal{L})$. We need to show that there is a map $[\neg\varphi] \leq [\alpha]$ in $\mathfrak{A}_i^{op}(\mathcal{L})$. It suffices to show that $\alpha \vdash \neg\varphi$:

$$\dfrac{\dfrac{\overline{\varphi \vdash (\alpha \to \bot) = \neg\alpha}}{\varphi, \alpha \vdash \bot} (\to r)^{-1}}{\alpha \vdash (\varphi \to \bot) = \neg\varphi} \to r$$

where a derivation of $\varphi \vdash \neg\alpha$ is given by assumption.

□

We can now look into the same situation as above in the case of classical logic. To achieve this purpose, we proceed in the same way as above: we consider a language $\mathcal{L}$ for the classical propositional logic CL and the ordered category $\mathfrak{A}_c(\mathcal{L})$ corresponding to the poset $(\mathfrak{A}_c(\mathcal{L}), \leq)$, as defined in Chapter ?!?!. As we did before, we define the **false functor** $\bot$, the **true functor** $\top$, the **conjuntion functor** $\wedge$, the **disjunction functor** $\vee$, the **implication functor** $\dashrightarrow$ and the **negation functor** $\neg$. Also, we assume the same notations as before, so we will denote the maps in the ordered category $\mathfrak{A}_c(\mathcal{L})$ by $\leq$.

Theorem 6.3. *Let $\mathcal{L}$ be a language for the classical propositional logic CL and let $\mathfrak{pr}$ be a fixed proposition. Then:*

1. *the true functor $\top$ is right adjoint to the trivial functor $!$:*

$$! \dashv \top$$

2. *the false functor $\bot$ is left adjoint to the trivial functor $!$:*

$$\bot \dashv !$$

3. *the conjunction functor $(-) \wedge (-)$ is right adjoint to the diagonal functor Δ:*

$$\Delta \dashv (-) \wedge (-)$$

4. *the disjunction functor $(-) \vee (-)$ is left adjoint to the diagonal functor Δ:*

$$(-) \vee (-) \dashv \Delta$$

5. *the implication of $\mathfrak{pr}$ functor $(-)^{\mathfrak{pr}}$ is right adjoint to the conjunction of $\mathfrak{pr}$ functor $(-) \wedge \mathfrak{pr}$:*

$$(-) \wedge \mathfrak{pr} \dashv (-)^{\mathfrak{pr}}$$

6. *the negation functor $\neg(-)$ is left adjoint to the opposite negation functor $\neg^{op}(-)$:*

$$\neg(-) \dashv \neg^{op}(-)$$

7. *the negation functor $\neg(-)$ is right adjoint to the opposite negation functor $\neg^{op}(-)$:*

$$\neg^{op}(-) \dashv \neg(-)$$

Proof. By Theorem (6.2), it suffices to show only the last point, i.e. $\neg^{op}(-) \dashv \neg(-)$.
Let $[\varphi] \in Ob(\mathfrak{A}_c(\mathcal{L}))$. There is a map $\neg^{op}(\neg([\varphi])) = [\neg\neg\varphi] \leq [\varphi]$ in $\mathfrak{A}_c(\mathcal{L})$. Indeed:

$$\dfrac{\dfrac{\overline{\varphi \vdash \varphi, \bot}}{\vdash \varphi \to \bot, \varphi} \to r \qquad \overline{\bot \vdash \varphi}}{(\varphi \to \bot) \to \bot = \neg\neg\varphi \vdash \varphi} \to l$$

Let $[\alpha] \in Ob(\mathfrak{A}_c^{op}(\mathcal{L}))$ and let $\neg^{op}([\alpha] = [\neg\alpha] \leq [\varphi]$ be a map in $\mathfrak{A}_c(\mathcal{L})$. Then we have a derivation

$$\dfrac{\begin{matrix}\vdots\, D \\ \vdash \alpha, \varphi\end{matrix} \qquad \overline{\bot \vdash \varphi}}{(\alpha \to \bot) = \neg\alpha \vdash \varphi} \to l$$

where D is a derivation. We need to find a map $[\alpha] \leq [\neg\varphi]$ in $\mathfrak{A}_c^{op}(\mathcal{L})$, thus we have to show that $\neg\varphi \vdash \alpha$:

$$\dfrac{\begin{matrix}\vdots\, D \\ \vdash \alpha, \varphi\end{matrix} \qquad \overline{\bot \vdash \alpha}}{(\varphi \to \bot) = \neg\varphi \vdash \alpha} \to l$$

where the derivation D is given by assumption. □

6.3 Heyting and Boolean algebras via adjunctions

We defined a Heyting algebra as a lattice $(X, \leq, \wedge, \vee)$ with least element 0 and greatest element 1, where an operator $\to: X \times X \longrightarrow X$ is defined and has the property: $\forall z \in X.(z \leq x \to y \Leftrightarrow z \wedge x \leq y)$. It follows that a Heyting algebra can as well be looked at as a poset X where the infimum $\wedge$ and the supremum $\vee$ of any two elements in X always exist, there are least and greatest elements and an operator $\to$ can be defined. However, all this can be formalized through categorical language, as a consequence of what has just been shown in the previous theorems.

However, the entire previous section was focused only on the Lindenbaum category $\mathfrak{A}(\mathcal{L})$ of the classical or intuitionistic logic. It is really easy, though, to extend the notion of $\top$, $\bot$, infimum $\wedge$, supremum $\vee$ and implication $\to$ to a generic preordered category (the negation $\neg$ can be defined using $\to$ and $\bot$, as we already know). In fact, we already proved that $\bot$ and $\top$ are the initial and final object, as well as that $\wedge$ and $\vee$ are the binary product and coproduct in $\mathfrak{A}(\mathcal{L})$. We do not know jet what the implication for an arbitrary preordered category stands for, but we can make sure that it exists in that category by checking the existence of a right adjoint to the product functor, as we proved in the previous section in the case of Lindenbaum categories for intuitionistic and classical logic. For now we denote the "general" implication functor for an object C as $(-)^C$, and we will make this notion clearer soon.

Proposition 6.1. *Let $\mathcal{H}$ be an ordered category. $\mathcal{H}$ is a Heyting algebra if and only if:*

- *the diagonal functor $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \times \mathcal{H}$ has right adjoint $(-) \wedge (-)$ and left adjoint $(-) \vee (-)$*

$$\Delta \dashv (-) \wedge (-) \text{ and } (-) \vee (-) \dashv \Delta$$

- *the trivial functor $! : \mathcal{H} \longrightarrow \mathbf{1}$ has right adjoint $\top$ and left adjoint $\bot$*

$$! \dashv \top \text{ and } \bot \dashv !$$

- *for every $A \in Ob(\mathcal{H})$ there exists an adjunction between the functors $(-) \wedge A : \mathcal{H} \longrightarrow \mathcal{H}$ on the left and $(-)^A : \mathcal{H} \longrightarrow \mathcal{H}$ on the right*

$$(-) \wedge A \dashv (-)^A$$

Proof. It follows directly from Theorem (6.2) and the remarks above. □

In a Heyting algebra we defined the negation $\neg$ such that for any element x of the Heyting algebra: $\neg x := x \to 0$, where 0 is the least element of the algebra. Theorem (6.2) claims that, given a category $\mathcal{H}$ that is a Heyting algebra, there is an adjunction between functors $\neg(-) : \mathcal{H} \longrightarrow \mathcal{H}^{op}$ on the left and $\neg^{op}(-) : \mathcal{H}^{op} \longrightarrow \mathcal{H}$ on the right:

$$\neg(-) \dashv \neg^{op}(-)$$

Theorem (6.3) shows that in the case of classical logic, so when we are dealing with an underlying Boolean algebra, there is also an adjunction

$$\neg^{op}(-) \dashv \neg(-)$$

Proposition 6.2. *Let $\mathcal{B}$ be a category and a Heyting algebra. Then $\mathcal{B}$ is a Boolean algebra if and only if $\neg^{op}(-) \dashv \neg(-)$, where $\neg^{op}(-) : \mathcal{B}^{op} \longrightarrow \mathcal{B}$ is an equivalence.*

Proof. We recall that a Heyting algebra in which $\neg\neg x = x$ for every element x is a Boolean algebra. If $\mathcal{B}$ is a Boolean algebra, then there is an isomorphism $\neg\neg A \cong A$ for every object $A \in Ob(\mathcal{B})$. It follows that $\neg^{op}(-) : \mathcal{B}^{op} \longrightarrow \mathcal{B}$ is an equivalence and therefore there are $\eta : 1_{\mathcal{B}^{op}} \longrightarrow \neg \circ \neg^{op}$ and $\varepsilon : \neg^{op} \circ \neg \longrightarrow 1_{\mathcal{B}}$ which are respectively the inverses of the counit and unit of the adjunction $\neg(-) \dashv \neg^{op}(-)$, and in particular they give also rise to an adjunction $\neg^{op}(-) \dashv \neg(-)$. Conversely, if $\neg^{op}(-) : \mathcal{B}^{op} \longrightarrow \mathcal{B}$ is an equivalence and there is an adjunction $\neg^{op}(-) \dashv \neg(-)$, then there is an isomorphism $\neg\neg A \cong A$ for every object $A \in Ob(\mathcal{B})$, and so the Heyting algebra $\mathcal{B}$ is in fact a Boolean algebra. □

Chapter 7

Logical quantifiers as adjoint functors

We need to introduce now cartesian closed categories. Their importance is twofold: on the one hand they permit to characterize in an easy elegant fashion Heyting (and thus Boolean) categories, on the other they are at the basis of the definition of hyperdoctrines. A hyperdoctrine is a certain category (together with an appropriate functor, to be precise) in which the process of logical quantification makes sense. Therefore, not only propositional logic can be treated categorically (i.e. by means of adjoint functors), but also predicate logic. And there is more to it than just that: thanks to hyperdoctrines, even predicate logic with equality can be merged into category theory.

7.1 Cartesian closed categories

Lawvere went on elaborating his program for the categorification of logic without respite all during the 1960s. In 1965 and 1966 he presented two papers at the *Meeting of the Association of Symbolic Logic*, which were published a year later ([Law66], [Law67]) and which contained the extension of the results obtained for algebraic theories to first-order theories in general. The notion of **elementary theory** was introduced and it was proved to correspond to some associated first-order theory. A categorical proof of the completeness theorem was sketched and it was showed that completeness amounts to the existence of specific adjoint functors. What is relevant, though, is that quantification was presented for the first time via adjoint functors. In particular, Lawvere showed how quantification arose naturally as an adjunction to the elementary operation of substitution, which is the basic natural operation of first-order logic, whereas in the classical view this operation is defined by recursion. On the top of that, Lawvere did not refer specifically to the category **Set** of sets and was in fact able to provide a point of view that generalized to arbitrary categories.

These two facts convinced many logicians and mathematicians that this was the right analysis of quantifiers, and categorical logic began to be regarded by many as a brand new branch of mathematics, whose master-plan for further developments was clearly stated and accepted as it was presented in three all-important publications in 1969 and 1970 ([Law69a], [Law69b] and [Law70a]).

In [Law69b] in particular, Lawvere stressed with more emphasis the pivotal role of adjunction, which became in his opinion the real primary concept. While in the previous paper he only used adjoint functors to show that some category was equivalent to some other fixed one or to establish specific properties of given functors in a context, on this occasion he even used them to define certain categories, namely **cartesian closed categories** (CC categories).

Lemma 7.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two ordered categories whose associated orders are sup-completed (i.e. there exists the supremum of every subset of the preorder), and let $F : \mathcal{P} \longrightarrow \mathcal{Q}$ be a functor between them. Then: F has a right adjoint if and only if F preserves arbitrary suprema.*

Proof. ($\Rightarrow$) Let's suppose that F has a right adjoint G, so $F \dashv G$. Let's take a subset $\{A_i \in Ob(\mathcal{P}|i \in I\}$ of objects of $\mathcal{P}$, whose supremum is P. Of course, $\{F(A_i)|i \in I\}$ is a subset of objects in $\mathcal{Q}$, and its supremum does exist by hypothesis. Let it be Q. Since $F \dashv G$, we have that $F(A_i) \leq Q \Leftrightarrow A_i \leq G(Q)$ for all $i \in I$, hence $P \leq G(Q)$ since P in the supremum of all A_i, $i \in I$, and by the condition of adjunction between F and G we get $F(P) \leq Q$. In addition, every map $A_i \leq P$ in $\mathcal{P}$ is sent by F into maps $F(A_i) \leq F(P)$ in $\mathcal{Q}$, and we should have that $Q \leq F(P)$ since Q is the supremum of all $F(A_i)$, $i \in I$. Concluding $F(P) = Q$.

($\Leftarrow$) Let's suppose that F preserves arbitrary suprema and let $B \in Ob(\mathcal{Q})$. The set $\{A \in Ob(\mathcal{P})|F(A) \leq B\}$ is a subset of $Ob(\mathcal{P})$, hence there exists its supremum, which we take as $G(B)$. Thanks to the hypotheses, it is straightforward to show that this definition gives rise to a functor $\mathcal{Q} \longrightarrow \mathcal{P}$, $B \longmapsto sup\{A \in Ob(\mathcal{P})|F(A) \leq B\}$. Also, it is easy to see that $A \leq G(B) \Leftrightarrow F(A) \leq B$ for every $A \in Ob(\mathcal{P})$ and $B \in Ob(\mathcal{Q})$, thus $F \dashv G$. $\square$

Let $\mathcal{C}$ be a category with binary products and let $C \in Ob(\mathcal{C})$. We can define a functor from $\mathcal{C}$ to itself as the product for a fixed object of the category, more precisely if $C \in Ob(\mathcal{C})$ is a fixed object of the category we call the **product functor with the object** C the following functor:

$$\begin{array}{cccc} (-)\times C: & \mathcal{C} & \longrightarrow & \mathcal{C} \\ & X & \longmapsto & X \times C \\ & (X \longrightarrow Y) & \longmapsto & (X \times C \longrightarrow Y \times C) \end{array}$$

Definition 7.1. *Let $\mathcal{C}$ be a category with binary products. It is said to be a* ***cartesian closed*** *category when, for every object $C \in Ob(\mathcal{C})$, the product functor $(-)\times C : \mathcal{C} \longrightarrow \mathcal{C}$ has a right adjoint $(-)^C : \mathcal{C} \longrightarrow \mathcal{C}$, sending each object $X \in Ob(\mathcal{C})$ into an object X^C called* ***space of the morphisms*** *from C to X.*

Example 7.1. The category **Set** is cartesian closed. To begin with, let's notice that for any two sets A and B the collection B^A of the maps from A to B is still a set. Let $S \in Ob(\mathbf{Set})$ be fixed. For any sets X, Y there is a natural isomorphism $\mathbf{Set}(X, Y^S) \cong \mathbf{Set}(X \times S, Y)$ defined in this way: any map $f : X \longrightarrow Y^S$ is sent into a map $\overline{f} : X \times S \longrightarrow Y$, $(x, s) \longmapsto (f(x))(s)$, and any map $g : X \times S \longrightarrow Y$ is sent into a map $\overline{g} : X \longrightarrow Y^S$, $x \longmapsto g(x, -)$. Consequently, $(-) \times S \dashv (-)^S$.

The first investigations about the role of CC categories in categorical logic were conducted by Lawvere and Lambek, who was bot in Zurich during the middle 1960s. No sooner had Lawvere found out the close relation between CC categories and Church's λ-calculus, than a lot of papers and publications on the topic began to appear. Lambek was the first to explore the relationships between categories and deductive systems. Together with Scott, he developed sistematically the connection between CC categories and typed λ-calculi and showed the equivalence between the category of typed λ-calculi and the category of CC categories ([LS86]).

Among the many scopes for which the class of CC categories is used for, we recall its role in the treatment of well-known diagonal arguments, such as those of Cantor, Russell, Gödel and Tarski: Lawvere used CC categories to present these arguments, after noticing that they have the same structure, which can take the form of a fixed point theorem based on the properties of CC categories.

More importantly, the cartesian closure can be used to characterize categories that are Heyting algebras. Let's consider a preordered cartesian closed category. Up to now, we have shown that preordered categories are extremely meaningful from a logical point of view, since inside them we can build logical structures by using the language of adjunctions. But there is a little bit more to say: cartesian closure can tell something about logical structures, too. Previously, we generalized the notions of $\top$, $\bot$, $\wedge$ and $\vee$ for an arbitrary preordered category. The deep meaning of implication in a generic preordered category was missing. We only knew that its existence was equivalent to the existence of a right adjoint to the functor $(-) \wedge C$, which we can interpret more generally as $(-) \times C$, for every object C. Definition (7.1) clarifies now that the space of morphisms is the generalization of the notion of implication.

Proposition 7.1. *Let $\mathcal{H}$ be an ordered category, Then $\mathcal{H}$ is a Heyting category if and only if $\mathcal{H}$ is cartesian closed and has finite coproducts.*

Proof. It follows trivially from what has just been asserted. □

Requiring a category to be cartesian closed can be somehow really strict. Often, a sort of weaker version of cartesian closure turns out to be far more handy. We recall a special case of comma categories, which are useful in order to deal with local cartesian closure: if $\mathcal{C}$ is a category and $C \in Ob(\mathcal{C})$, an object (X, f) of the **slice category** $\mathcal{C}/C$ is a map $f : X \longrightarrow C$ with $X \in Ob(\mathcal{C})$ and a map $(X, f) \longrightarrow (X', f')$ in $\mathcal{C}/C$ is a map $g : X \longrightarrow X'$ such that $f' \circ g = f$.

Definition 7.2. *A category $\mathcal{C}$ is a* ***locally cartesian closed category*** *if its slice categories $\mathcal{C}/C$ are cartesian closed, for every $C \in Ob(\mathcal{C})$.*

It can be shown that a locally cartesian closed category with a terminal object is a cartesian closed category itself.

7.2 Hyperdoctrines

All the three 1969-70 papers [Law69a], [Law69b] and [Law70a] sketched extensions of Lawvere's earlier work on algebraic theories to higher-order and especially first-order theory. He formalized the proof of logical quantifiers as adjoint functors to substitution, proved that the comprehension principle can be presented via adjunction and sketched how a category can be constructed from a given higher-order logical theory. Besides that, most papers on categorical logic of the period had a crucial common moral, namely the correspondence between propositions of a deductive system with the objects of a corresponding category and the equivalent classes of proofs of a deductive system with the maps of a corresponding category. This was summarized by the slogans *propositions-as-objects* and *proofs-as-morphisms.* During the 1970s, by looking at the upshots of the first decade of categorical logic, Lawvere realized that category theory was not a whatever language for mathematics, but rather a *formal* language for mathematics.

The problem to formalize in categoric-theoretic terms quantification, equality and comprehension schema for a logical language was to Lawvere far more challenging than making explicit the adjoint-functorial nature of connectives. The point was that quantification needed an organic and specific structure in which to live and Lawvere himself began investigating into the nature of such an entity with no starting points but his intuition. Specifically, Lawvere wanted to be able to analyze quantification, equality and the comprehension schema as adjoints to elementary functors, including them so in the algebraic approach. His plan was to organize the logical data in a complicated but really powerful structure, which he called a **hyperdoctrine**.

analysis of the comprehension principle and Bénabou's work on fibred category.

As regards Lawvere's contribution, he had been interested in the extensions of the connections between CC categories and logic in the direction of higher-order logic and type theory since the New York conference on applications of categorical algebra in 1968. Specifically, Lawvere wanted to be able to analyze equality and the comprehension schema as adjoints to elementary functors, including them so in the algebraic approach. His plan was to organize the logical data in a complicated but really powerful structure, which he called a **hyperdoctrine**. This new concept did its very first historical appearance in 1969 paper [Law69a], where adjunction was proved to be sufficient to deal with quantification, and was later improved in 1970 paper [Law70a], where Lawvere presented equality and comprehension shema in hyperdoctrines. Here is the very first historical definition of hyperdoctrine.

Definition 7.3. *A* ***hyperdoctrine*** *consists of*

1. *a cartesian closed category* $\mathbf{T}$*. The objects of* $\mathbf{T}$ *are called* ***types****, its maps are called* ***terms*** *and those ones of the form* $1 \longrightarrow C$ *are called* ***constant terms*** *of type* C*. Terminal object, product and exponentiation by* C *(with* C *a type in* $\mathbf{T}$*) are respectively denoted by:*

$$1,\ \times\ ,\ ()^C$$

2. *for every type C, a corresponding cartesian closed category* $\mathbf{P(C)}$ *called the category of* ***attributes*** *of type C. The maps between attributes of type C are called* ***deductions*** *over C. Terminal object, product and exponentiation by φ (with φ an attribute of type C) are respectively denoted by:*
$$1_C,\ \wedge_C\ ,\ \varphi_C \Rightarrow ()$$
In particular, given φ and ψ attributes of type C, the evaluation over ψ along φ is given by the map $\varphi \wedge_C (\varphi \Rightarrow \psi) \longrightarrow \psi$ in $\mathbf{P(C)}$*, so it is called a* ***modus ponens deduction****.*

3. *for each term $f : X \longrightarrow Y$ in* $\mathbf{T}$*, a corresponding functor $[f/-]$ called* ***substitution****, defined by:*
$$\begin{array}{rccc} [f/-]: & P(Y) & \longrightarrow & P(X) \\ & \psi & \longmapsto & [f/\psi] \\ & (\varphi \longrightarrow \psi) & \longmapsto & ([f/\varphi] \longrightarrow [f/\psi]) \end{array}$$
where $[f/\psi]$ is the attribute of type X resulting from substituting f in ψ. If $g : Y \longrightarrow Z$, we assume that $[f/-] \circ [g/-] = [(g \circ f)/-]$.

4. *for each term $f : X \longrightarrow Y$, two functors $\Sigma f(-)$ and $\Pi f()$*
$$\Sigma f(-) : P(Y) \longrightarrow P(X) \text{ and } \Pi f(-) : P(Y) \longrightarrow P(X)$$
such that they are respectively left and right adjoints to substitution f:
$$\Sigma f(-) \dashv [f/-] \dashv \Pi f(-).$$
For any attribute φ of type X, we call $\Sigma f(\varphi)$ the ***existential quantification*** *of φ along f and we call $\Pi f(\varphi)$ the* ***universal quantification*** *of φ along f. From $\Sigma f(-) \dashv [f/-]$ the existence of two natural isomorphisms follows, one between deductions $\Sigma f(\varphi) \longrightarrow \psi$ over X and deductions $\varphi \longrightarrow [f/\psi]$ over Y, and the other one between deductions $\psi \longrightarrow \Pi f(\varphi)$ over Y and deductions $[f/\psi] \longrightarrow \varphi$ over X, for every attribute φ of type X and every attribute ψ of type Y. Finally, by the functoriality of the substitution: $\Sigma(g \circ f)(\varphi) = \Sigma g(\Sigma f(\varphi))$ and $\Pi(g \circ f)(\varphi) = \Pi g(\Pi f(\varphi))$ for every term $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ and every deduction φ over X.*

7.2.1 Hyperdoctrines for predicate logic and for predicate logic with equality

The idea beyond the notion of a hyperdoctrine is the axiomatization of the collection of slices of a locally cartesian closed category. More precisely, a hyperdoctrine is given by a category together with a functor sending its objects into slice-like categories, satisfying some reasonable conditions.

As far as we are concerned, a hyperdoctrine will be thought of as a collection of contexts together with the operations of contexts extension/substitution and quantification on the category of propositions in each contexts. So, putting things under a certain perspective, giving a hyperdoctrine over some category amounts to equipping it with a specific kind of logic. Specifically, a hyperdoctrine on a category $\mathcal{C}$ for a chosen logic L is a functor
$$\mathfrak{D} : \mathcal{C}^{op} \longrightarrow \mathbf{L}$$
to some 2-category $\mathbf{L}$, whose objects are categories whose internal logic corresponds to L. In this way, every object of $\mathcal{C}$ is endowed with an L-logic. We will take L to be the propositional logic and $\mathbf{L}$ a 2-category of posets, for instance Heyting or Boolean algebras.

A 2-category is a generalization of the notion of category, where on top of objects and maps, there are further structures called 2-morphisms, which connect a map to another one. While maps can be composed along the objects, the 2-morphisms can be composed both along objects (this is usually called horizontal composition) and along maps (this is usually called vertical composition). The most important example of 2-category is $\mathbf{Cat}$, where objects are categories, maps are functors and 2-morphisms are natural trasformation. Quite easily, we can also think of **Ha** and **Ba** as 2-categories.

A hyperdoctrine is then an incarnation of first-order predicate logic. With this idea borne in mind, we will also require that for every map f in $\mathcal{C}$ the map $\mathfrak{D}(f)$ in $\mathbf{L}$ has both left and right adjoints, satisfying the so called Beck-Chevalley conditions and Frobenius reciprocity. These adjunctions are regarded as the action of the existential and universal quantifiers along the map f.

Concluding, a hyperdoctrine can also be viewed as a categorical way of adding quantifiers to a given kind of logic, and so it becomes the necessary tool for our purpose to pass from propositional to predicate logic.

We consider now the intuitionistic case. The classical case can be seen as a natural extension of what we we are going to present in intuitionistic logic.

Definition 7.4. *A* ***hyperdoctrine for the predicative intuitionistic logic*** *IL is a functor*

$$\mathfrak{D} : \mathcal{C}^{op} \longrightarrow \mathbf{Ha}$$

such that:

(i) $\mathcal{C}$ is a category with finite products, called the ***base*** *of the hyperdoctrine, and for every $X \in Ob(\mathcal{C})$ the category $\mathfrak{D}(X)$ is called a* ***fiber*** *on X (or a predicate over X), objects $X \in Ob(\mathcal{C})$ are often called a* ***types*** *and maps in $\mathcal{C}$ are often called* ***terms****;*

(ii) for every projection $\pi_i : X_1 \times X_2 \longrightarrow X_i$ $(i = 1,2)$ in $\mathcal{C}$, the functor $\mathfrak{D}(\pi_i) : \mathfrak{D}(X_i) \longrightarrow \mathfrak{D}(X_1 \times X_2)$ has a left adjoint $\exists_{\pi_i} : \mathfrak{D}(X_1 \times X_2) \longrightarrow \mathfrak{D}(X_i)$

$$\exists_{\pi_i} \dashv \mathfrak{D}(\pi_i)$$

satisfying the Beck-Chevalley condition and the Frobenius reciprocity;

(iii) for every projection $\pi_i : X_1 \times X_2 \longrightarrow X_i$ $(i = 1,2)$ in $\mathcal{C}$, the functor $\mathfrak{D}(\pi_i) : \mathfrak{D}(X_i) \longrightarrow \mathfrak{D}(X_1 \times X_2)$ has a right adjoint $\forall_{\pi_i} : \mathfrak{D}(X_1 \times X_2) \longrightarrow \mathfrak{D}(X_i)$

$$\mathfrak{D}(\pi_i) \dashv \forall_{\pi_i}$$

satisfying the Beck-Chevalley condition.

The ***Beck-Chevalley condition*** *for $\exists$ states the following: given any two projections $\pi : X \longrightarrow A$ and $\pi' : X' \longrightarrow A'$, for every pullback diagram*

$$\begin{tikzcd} X' \arrow[r, "\pi'"] \arrow[d, "f'"'] & A' \arrow[d, "f"] \\ X \arrow[r, "\pi"'] & A \end{tikzcd}$$

for every $\beta \in \mathfrak{D}(X)$ the canonical map

$$\exists_{\pi'} \mathfrak{D}(f')(\beta) \leq \mathfrak{D}(f) \exists_{\pi}(\beta)$$

in $\mathfrak{D}(A')$ is an isomorphism.

The ***Frobenius reciprocity*** *states that for every projection $\pi : X \longrightarrow A$ and for all $\alpha \in D(A)$, $\beta \in \mathfrak{D}(X)$, the canonical map*

$$\exists_{\pi}(\mathfrak{D}(\pi)(\alpha) \wedge_X \beta) \leq \alpha \wedge_A \exists_{\pi}(\beta)$$

in $\mathfrak{D}(A)$ is an isomorphism.

*The **Beck-Chevalley condition** for $\forall$ states the following: given any two projections $\pi : X \longrightarrow A$ and $\pi' : X' \longrightarrow A'$, for every pullback diagram*

$$\begin{tikzcd} X' \arrow[r, "\pi'"] \arrow[d, "f'"'] & A' \arrow[d, "f"] \\ X \arrow[r, "\pi"'] & A \end{tikzcd}$$

for every $\beta \in \mathfrak{D}(X)$ the canonical map

$$\mathfrak{D}(f)\forall_\pi(\beta) \leq \forall_{\pi'}\mathfrak{D}(f')(\beta)$$

in $\mathfrak{D}(A')$ is an isomorphism.

This structure is often called a **first-order hyperdoctrine**. One may wonder if for each term $X \in Ob(\mathcal{C})$ an equality predicate can be defined by

$$\exists_{\delta_X}(\top_{\mathfrak{D}(X)}) \in \mathfrak{D}(X \times X)$$

where $\top_{\mathfrak{D}(X)}$ is the top element of the Heyting algebra $\mathfrak{D}(X)$ and $\delta_X : X \times X \longrightarrow X$ in $\mathcal{C}$. However, this is not enough to define a good notion of equality, and we need to assume a little more, in order to ensure a sound interpretation of first order logic with equality in the hyperdoctrine. The turning idea which allows a good definition of equality in hyperdoctrines can be suggested by the notions of **fibred category** and **fibration**, which were the starting point of Benabou's work during the 1970s.

Before giving a definition of fibration, let's introduce the notation $\begin{tikzcd} \mathcal{C} \arrow[d, "p"] \\ \mathcal{D} \end{tikzcd}$ for the functor $p : \mathcal{C} \longrightarrow \mathcal{D}$. Given $X \in Ob(\mathcal{D})$, the **fibre** $p^{-1}(X)$ over X is the category whose objects are those $A \in Ob(\mathcal{C})$ such that $p(A) = X$ and whose maps are those $f : A \longrightarrow B$ in $\mathcal{C}$ such that $p(f) = 1_X$ in $\mathcal{D}$. $\mathcal{C}$ is called the **base category** and $\mathcal{D}$ is called the **total category**. An object $A \in Ob(\mathcal{C})$ such that $p(A) = X$ is said to be **above** X and a map f in $\mathcal{C}$ such that $p(f) = u$ is said to be **above** u.

Definition 7.5. *Let $p : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.*
*A map $f : C \longrightarrow B$ in $\mathcal{C}$ is said to be **cartesian** over $u : X \longrightarrow Y$ in $\mathcal{D}$ if $p(f) = u$ and if for every $g : C \longrightarrow B$ such that $p(g) = u \circ w$ for some $w : p(C) \longrightarrow X$ in $\mathcal{D}$ then there is a unique $h : C \longrightarrow A$ in $\mathcal{C}$ above w such that $f \circ h = g$.*

$$\begin{tikzcd} C \arrow[d, dashed, "h"'] \arrow[dr, "g"] & \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} p(C) \arrow[d, "w"'] \arrow[dr, "u \circ w = p(g)"] & \\ X \arrow[r, "u"'] & Y \end{tikzcd}$$

*The functor $p : \mathcal{C} \longrightarrow \mathcal{D}$ is a **fibration** or a **fibred category** if for every $B \in Ob(\mathcal{C})$ and $u : X \longrightarrow p(B)$ in $\mathcal{D}$, there is a cartesian map $f : A \longrightarrow B$ in $\mathcal{C}$ above u.*

Bénabou gave condition for a fibration to be small or locally small and showed that the concept of fibration was what was needed to deal with foundational issues from a general point of view: up to that day, category theorists had assumed that being small or locally small had a meaning in the ambient universe in which they were working, so they had assumed a category of sets for foundational purposes, in particular for issues of size.

In the 1980s, category logicians combined Lawvere's notion of hyperdoctrine with Bénabou's groundwork in order to provide a semantical framework for type theories and so completeness theorems for them (e.g. [Jac99]). Lawvere himself found his inspiration for the definition of equality and comprehension schema in a hyperdoctrine right in fibrations.

Definition 7.6. *A* ***hyperdoctrine for the predicative intuitionistic logic with equality*** *$IL_=$ is a hyperdoctrine for IL*

$$\mathfrak{D} : \mathcal{C}^{op} \longrightarrow \mathbf{Ha}$$

satisfying in addition the following condition: for every $A \in Ob(\mathcal{C})$ there exists $\delta_A \in Ob(\mathfrak{D}(A \times A))$, called ***fibred equality on*** *A such that, for every $X \in Ob(\mathcal{C})$ and for every map*

$$(\pi_1, \pi_2, \pi_2) : X \times A \longrightarrow X \times A \times A$$

in $\mathcal{C}$, the functor

$$\begin{array}{rccc} \exists_{\pi_1,\pi_2,\pi_2} : & \mathfrak{D}(X \times A) & \longrightarrow & \mathfrak{D}(X \times A \times A) \\ & \alpha & \longmapsto & \mathfrak{D}(1_X \times \pi_1)(\alpha) \wedge \mathfrak{D}((\pi_2, \pi_2))(\delta_A) \end{array}$$

is left adjoint to the functor

$$\mathfrak{D}(\pi_1, \pi_2, \pi_2) : \mathfrak{D}(X \times A \times A) \longrightarrow \mathfrak{D}(X \times A)$$

In symbols:

$$\exists_{\pi_1,\pi_2,\pi_2} \dashv \mathfrak{D}(\pi_1, \pi_2, \pi_2)$$

We can define in a similar way the notions of hyperdoctrine for predicative classical logic and of hyperdoctrine for predicative classical logic with equality, by substituting the category **Ha** for the category **Ba** in the definitions above.

The notion of hyperdoctrine permits to interpret multi-sorted first-order (either intuitionistic or classical) logic with equality: we intend the objects $X \in Ob(\mathcal{C})$ to be sorts and the maps in $\mathcal{C}$ to be terms, and so the functor $\mathfrak{D}$ assigns to each sort the Lindenbaum algebra of predicates upon that sort and to each term the operation of substitution of that term into predicates, left and right adjoints $\exists_-$ and $\forall_-$ to images of projections in $\mathcal{C}$ through $\mathfrak{D}$ provide existential and universal quantifications, the existence of further adjoints provides the possibility of interpreting equality, Beck-Chevalley conditions and Frobenious identity ensure that quantification commutes with substitution in the same appropriate way propositional quantifiers do.

The paper *Adjointness in Foundations* ([Law69a]) is, in addition to all the aspects reported so far, of particular philosophical and foundational interest too. In this publication Lawvere identified two aspects, namely the **Formal** and the **Conceptual**, which appeared to obey a general duality inherent to mathematics and this framework of duality constitued the scaffolding of categorical logic as it developed afterwords. The **Formal** was identified with the manipulation of symbols, while the **Conceptual** is identified with their content. Though it might sound really similar to that one between syntax and semantics, this distinction lay on a more general level: also foundations actually partook of the duality, being itself part of mathematics. Of course the syntax of logical systems belonged to the **Formal** and the semantics to the **Conceptual**. To the latter belonged also categorical logic. The duality between these two aspects was established by Lawvere by means of adjoint functors. He claimed indeed that there was an adjunction

$$\mathbf{Theories}^{op} \underset{structure}{\overset{semantics}{\rightleftarrows}} \mathbf{Conceptual}$$

sending each theory into the category of its models[1]. Also Lawvere suggested that there should be another adjunction

$$\mathbf{Formal} \rightleftarrows \mathbf{Theories}$$

describing "the presentation of the invariant theories by means of the formalized language appropriate to the species" ([Law69a], pg 295). By composition Lawvere obtained a family of adjoint functors

$$\mathbf{Formal}^{op} \rightleftarrows \mathbf{Conceptual}$$

[1]Given a theory $\mathbb{T}$, a model of $\mathbb{T}$ is a functor $\mathbb{T} \longrightarrow \mathbf{Set}$. The category of model of $\mathbb{T}$ is then a subcategory of the functor category $\mathbf{Set}^{\mathbb{T}}$.

This totally programmatic project have had concrete outcomes[2] since it lead to the development of **categorical doctrines** ([KR79]).

Categorical logic was not the only attempt at developing an algebraic framework for logic during those years and it is worth contrasting one with each other. During the 1970s, Joyal and Reyes ([JR]) argued the advantages of a category-theoretic approach over the others: first of all, the concept of category springs up in all branches of mathematics and seems really natural to a mathematician; also, certain categories used in different fields of mathematics are actually theories and consider them as such is fruitful; on top of that, many constructions used in model theory are nothing but specializations of general categorical constructions. Therefore, towards the end of 1960s and early 1970s, the following "dictionary" was elaborated and most logicians accepted it fully:

Logic	**Category Theory**
many-sorted theory	category
sort	object of a category
sorted formula	subobject
sorted term	map in a category
interpretation	functor
set-theoretical model	functor into **Set**
homomorphism	natural transformation

7.2.2 An example: the syntactic hyperdoctrine

The structure of hyperdoctrine behaves well also at the syntactic level, and this is crucial in our treatment of quantifiers. In order to define a syntactic hyperdoctrine, we need to define a contravariant functor from a category that can express syntatic facts either to the category **Ha** of Heyting algebras or the category **Ba** of Boolean algebras.

Definition 7.7. *Let $\mathcal{L}$ be a language (either for $IL_=$ or $CL_=$). We introduce a new category* **Cont** *and we call it the* ***category of contexts***. *Its objects are lists of distinct variables $(x_{j_1},...,x_{j_n})$; a map $\vec{t} := [t_1/x_{k_1},...,t_n/x_{k_m}] : (x_{j_1},...,x_{j_n}) \longrightarrow (x_{k_1},...,x_{k_m})$ in* **Cont** *is a list of term substitutions in $\mathcal{L}$, where the free variables of t_i are among $x_{j_1},...,x_{j_n}$, for all $i = 1,...,m$.*

It can be easily seen that composition in **Cont** is well-defined and acts as a simultaneous substitution. Indeed, if $\vec{t} := [t_i/x_{k_i}]_{i=1,...,m} : (x_{j_1},...,x_{j_n}) \longrightarrow (x_{k_1},...,x_{k_m})$ and $\vec{s} := [s_i/x_{h_i}]_{i=1,...,l} : (x_{k_1},...,x_{k_m}) \longrightarrow (x_{h_1},...,x_{h_l})$ are two maps in **Cont**, then their composition is the map

$$\vec{s} \circ \vec{t} := [s_i(t_1/x_{k_1},...,t_m/x_{k_m})/x_{h_i}]_{i=1,...,l} : (x_{j_1},...,x_{j_n}) \longrightarrow (x_{h_1},...,x_{h_l})$$

which is still in **Cont** and each term $s_i(\vec{t}/\vec{x}_k)/x_{h_i}$ has free variables among $x_{j_1},...,x_{j_n}$.

Given $\vec{t} := [t_i/x_{k_i}]_{i=1,...,m}$ and $\vec{s} := [s_i/x_{h_i}]_{i=1,...,m}$ two maps in **Cont**, we define equality on maps by:

$$\vec{t} = \vec{s} \text{ if and only if } t_i = s_i \text{ for all } i \in \{1,...,m\}$$

Composition in **Cont** is associative: given

$$\vec{t} := [t_i/x_{k_i}]_{i=1,...,m} : (x_{j_1},...,x_{j_n}) \longrightarrow (x_{k_1},...,x_{k_m})$$

$$\vec{s} := [s_i/x_{h_i}]_{i=1,...,l} : (x_{k_1},...,x_{k_m}) \longrightarrow (x_{h_1},...,x_{h_l})$$

$$\vec{u} := [u_i/x_{g_i}]_{i=1,...,p} : (x_{h_1},...,x_{h_l}) \longrightarrow (x_{g_1},...,x_{g_p})$$

[2]It is worth mentioning that this is not an unanimous opinion. For instance, Corry asserted that "this proposal [...] remained at the programmatic level and no one seems to have developed it further" ([Cor96]).

maps in **Cont**,we have that $u_i[\vec{s}/\vec{x}_h][\vec{t}/\vec{x}_k] = u_i[\vec{s}[\vec{t}/\vec{x}_k]/\vec{x}_h]$ for all $i \in \{1,...,p\}$ since $x_{k_1},...,x_{k_m}$ don't appear in u_i.

For every list of variables $(x_{j_1},...,x_{j_n})$ there is the identity map

$$\vec{x}_j[x_{j_i}/x_{j_i}]_{i=1,...,n} : (x_{j_1},...,x_{j_n}) \longrightarrow (x_{j_1},...,x_{j_n})$$

as it is easy to verify.

Furthermore, $\mathcal{C}$ has the empty list $(\varnothing)$ as terminal object: a map in **Cont** from any list of distinct variables to the empty list simply deletes all dependences on the variables of that list.

The category **Cont** has all binary products: given $(x_{j_1},...,x_{j_n})$ and $(x_{k_1},...,x_{k_m})$, their product is the list

$$(x_{j_1},...,x_{j_n},x_{J+k_1},...,x_{J+k_m}) \text{ with } J = max\{j_1,...,j_n\}$$

and the projections are given by

$$\vec{x}_j := [\vec{x}_j/\vec{x}_j, \vec{x}_k/\varnothing] : (x_{j_1},...,x_{j_n},x_{J+k_1},...,x_{J+k_m}) \longrightarrow (x_{j_1},...,x_{j_n})$$

$$\vec{x}_k := [\vec{x}_j/\varnothing, \vec{x}_k/\vec{x}_k] : (x_{j_1},...,x_{j_n},x_{J+k_1},...,x_{J+k_m}) \longrightarrow (x_{j_1},...,x_{j_n})$$

Indeed, let $(x_{h_1},...,x_{h_l}) \in Ob(\mathbf{Cont})$ and let $\vec{t} : (x_{h_1},...,x_{h_l}) \longrightarrow (x_{j_1},...,x_{j_n})$ and $\vec{s} : (x_{h_1},...,x_{h_l}) \longrightarrow (x_{k_1},...,x_{k_m})$ be maps in **Cont**; then the map

$$\langle \vec{t}, \vec{s} \rangle : (x_{h_1},...,x_{h_l}) \longrightarrow (x_{j_1},...,x_{j_n},x_{J+k_1},...,x_{J+k_m})$$

is such that

$$\vec{x}_j \circ \langle \vec{t}, \vec{s} \rangle = \vec{t} \text{ and } \vec{x}_k \circ \langle \vec{t}, \vec{s} \rangle = \vec{s}$$

and it is also unique: if $\vec{u} : (x_{h_1},...,x_{h_l}) \longrightarrow (x_{j_1},...,x_{j_n},x_{J+k_1},...,x_{J+k_m})$ is such that $\vec{x}_j \circ \vec{u} = \vec{t}$ and $\vec{x}_k \circ \vec{u} = \vec{s}$, then $u_i = x_{j_i}$ for every $i \in \{1,...,n\}$ and $u_{J+k_i} = x_{J+k_i}$ for all $i \in \{1,...,m\}$, hence $\langle \vec{t}, \vec{s} \rangle = \vec{u}$.

Remark 7.1. Of course, we can generalize the just given definition of binary product to the definition of finite product. In particular, given a list of distinct variables $(x_{j_1},...,x_{j_n})$, we can as well define the object $(x_{j_1},...,x_{j_n})^n$ in **Cont**, as the product of $(x_{j_1},...,x_{j_n})$ with itself n times, with n a natural number. It can be noticed that the objects of the category of contexts **Cont** form a finitely generated set: $Ob(\mathbf{Cont}) = \{x_1^n | n \in \mathbb{N}\}$, proceeding by induction on the length of the list of distinct variables which give an object of **Cont**.

From now on, we will give different names to variables, meaning implicitly that we can anyway give them an order. We need to show now that the category **Cont** of ontects has the structure of a hyperdoctrine. More properly: given a language $\mathcal{L}$ for intuitionistic logic $IL_=$, there is a functor

$$H_{\mathcal{L}} : \mathbf{Cont}^{op} \longrightarrow \mathbf{Ha}$$

which satisfies the conditions of the definition of a hyperdoctrine and which we will call the intuitionistic **syntactic hyperdoctrine** $H_{\mathcal{L}}$ for the language $\mathcal{L}$. Of course, this can be naturally extended to the classical case. The natural way to define such a functor is exactly what does the job, as we will see. Thus we introduce the following definition, and then we will prove that it is a good one.

Definition 7.8. *Let $\mathcal{L}$ be a language for intuitionistic logic $IL_=$. The* ***intuitionistic syntactic hyperdoctrine*** *$H_{\mathcal{L}}$ for the language $\mathcal{L}$ is a functor*

$$H_{\mathcal{L}} : \mathbf{Cont}^{op} \longrightarrow \mathbf{Ha}$$

where:

- *for each object* $\vec{x} = (x_1, ..., x_n) \in Ob(\mathbf{Cont})$, *we define*

$$H_{\mathcal{L}} = \mathfrak{A}(\mathcal{L})(\vec{x}) := \{[\varphi(x_1, ..., x_n)] | \varphi \in Form_{\mathcal{L}}\}$$

  to be the Lindenbaum-Tarski algebra of the formulas with free variables in $\vec{x}$, *called the* ***fiber on variables*** $\vec{x}$;

- *for each map* $\vec{t} = [t_1/y_1, ..., t_m/y_m] : (x_1, ..., x_n) \longrightarrow (y_1, ..., y_m)$ *in* $\mathbf{Cont}$, *we define*

$$H(\vec{t}) : \mathfrak{A}(\mathcal{L})(y_1, ..., y_m) \longrightarrow \mathfrak{A}(\mathcal{L})(x_1, ..., x_n)$$

  to be the map in $\mathbf{Ha}$, *that is a Heyting algebra morphism, sending any formula* $\varphi(y_1, ..., y_m)$ *with free variables among* $y_1, ..., y_m$ *into the formula* $\varphi(y_1, ..., y_m)[y_1/t_1, ..., y_m/t_m]$ *with free variables among* $x_1, ..., x_n$.

Let $\mathcal{L}$ *be a language for classical logic* $CL_=$. *The* ***classical syntactic hyperdoctrine*** $H_{\mathcal{L}}$ *for the language* $\mathcal{L}$ *is a functor*

$$H_{\mathcal{L}} : \mathbf{Cont}^{op} \longrightarrow \mathbf{Ba}$$

defined in the same way as above.

In particular, if $\vec{x}$ and $\vec{x}'$ are lists of variables and $\pi : \vec{x} \longrightarrow \vec{x}'$ is a projection, we call $H_{\mathcal{L}}(\pi) : \mathfrak{A}(\mathcal{L})(\vec{x}') \longrightarrow \mathfrak{A}(\mathcal{L})(\vec{x})$ a **weakening functor**. It sends formulas with free variables into formulas with more extra free variables.

For reasons of clarity, we will consider objects in $Ob(\mathbf{Cont})$ of the form (x, y), instead of a more general case $(x_1, ..., x_n)$, though it is obvious that the generalization is just a matter of formality.

Let's consider $\vec{x_1}$, $\vec{x_2}$ lists of variables and $\pi_i : \vec{x_1} \times \vec{x_2} \longrightarrow \vec{x_i}$ a projection. By the definition of hyperdoctrine, there must be a left adjoint $\exists_{\pi_i} : \mathfrak{A}(\mathcal{L})(\vec{x_1} \times \vec{x_2}) \longrightarrow \mathfrak{A}(\mathcal{L})(\vec{x_i})$ to the weakening functor $H_{\mathcal{L}}(\pi_i)$. The functor $\exists_{\pi_i}$ sends every formula in $\mathfrak{A}(\mathcal{L})(\vec{x_1} \times \vec{x_2})$, thus every formula with free variables in $(\vec{x_1}, \vec{x_2})$, into the equivalent class of the existential quantification on the variables that are not involved in the substitution given by π_i. For instance, let's reduce (as we preannounced before) to the case of the projection $\pi : (x, y) \longrightarrow (x)$ and a formula $\psi = \psi(x, y)$ with free variables among x, y, so $[\psi] \in \mathfrak{A}(\mathcal{L})(x, y)$. For the sake of plainness, we will write the projection π as $\pi = [x/z] : (x, y) \longrightarrow (z)$: indeed, we know that changing the name of free variables in a formula does not change the formula itself (provided the new variables do not already appear in the formula). We need an adjunction $\exists_\pi(-) = \exists y.(-)[x/z] \dashv H_{\mathcal{L}}(\pi)$, so for any other formula $[\alpha] = [\alpha(z)] \in \mathfrak{A}(\mathcal{L})(z)$ with free variable at most z we should have that:

$$[\exists y.\psi[x/z]] \leq [\alpha] \Leftrightarrow [\psi] \leq H_{\mathcal{L}}(\pi)([\alpha])$$

but α does not depend on y, hence $H_{\mathcal{L}}(\pi)([\alpha]) = [\alpha]$ and the adjunction is equivalent to the following claim:

$$[\exists y.\psi[x/z]] \leq [\alpha] \Leftrightarrow [\psi] \leq [\alpha]$$

and this is exactly how it should be. Hence generalizing:

$$\exists_{\pi_i} \vdash H_{\mathcal{L}}(\pi_i)$$

The Beck-Chevalley condition for the functor $\exists$ turns out to rule how to substitute a free variable in an existential quantification. Let's consider again a projection $[x/x] : (x, y) \longrightarrow (x)$ in $\mathbf{Cont}$ and a formula $[\psi] = [\psi(x, y)] \in \mathfrak{A}(\mathcal{L})(x, y)$ as before. For every map $[\vec{w}/x] : (w_1, ..., w_n) \longrightarrow x$ in $\mathbf{Cont}$, the diagram

$$\begin{tikzcd}
(w_1, ..., w_n, y) \arrow[r, "{[\vec{w}/\vec{w}]}"] \arrow[d, "{[\vec{w}/x, y/y]}"'] & (w_1, ..., w_n) \arrow[d, "{[\vec{w}/x]}"] \\
(x, y) \arrow[r, "{[x/x]}"'] & (x)
\end{tikzcd}$$

is a pullback diagram: first of all $[x/x] \circ [\vec{w}/x, y/y] = [\vec{w}/x] = [\vec{w}/x] \circ [\vec{w}/\vec{w}]$, so the diagram above commutes; let $(u_1, ..., u_m) \in Ob(\mathbf{Cont})$ and let

$$\begin{array}{ccc} (u_1, ..., u_m) & \xrightarrow{f} & (w_1, ..., w_n) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle [\vec{w}/x]} \\ (x, y) & \xrightarrow[{[x/x]}]{\qquad} & (x) \end{array}$$

be a commutative diagram, then the unique map that makes the diagram above a pullback is $[f, y/y] : (u_1, ..., u_m) \longrightarrow (w_1, ..., w_n, y)$. We would like that the following holds:

$$[\exists y.(\psi[x/\vec{w}, y/y])] = [(\exists y.\psi)[x/\vec{w}]]$$

so in particular we need the map

$$\exists_{\vec{w}} H_{\mathcal{L}}([\vec{w}/x, y/y])([\psi]) \le H_{\mathcal{L}}([\vec{w}/x])\exists_{[x/x]}([\psi])$$

to be an isomorphism, that is exactly what the Beck-Chevalley condition claims. As we noticed some lines above, sometimes it turns out to be convenient or somehow clearer to write a projection $(x, y) \longrightarrow (x)$ by using a change of variables $(x, y) \longrightarrow (z)$, without affecting the meaning of formulas. It is easy to see that the Beck-Chevalley condition for such a notation results in the isomorphism:

$$[\exists y.(\psi[x/\vec{w}, y/y])] = [(\exists y.\psi[x/z])[z/\vec{w}]]$$

We would also ask that if β is a formula that does not depend on y, then $[\exists y.(\alpha(y) \wedge \beta)] \Leftrightarrow [\exists y.\alpha(y) \wedge \beta]$. This is nothing but the Frobenius reciprocity condition, which thus holds.

Let's consider again $\vec{x_1}$, $\vec{x_2}$ lists of variables and $\pi_i : \vec{x_1} \times \vec{x_2} \longrightarrow \vec{x_i}$ a projection. There must be a right adjoint $\forall_{\pi_i} : \mathfrak{A}(\mathcal{L})(\vec{x_1} \times \vec{x_2}) \longrightarrow \mathfrak{A}(\mathcal{L})(\vec{x_i})$ to the weakening functor $D(\pi_i)$. The functor $\forall_{\pi_i}$ sends every formula in $D(\vec{x_1} \times \vec{x_2})$, thus every formula with free variables in $(\vec{x_1}, \vec{x_2})$, into the equivalent class of the universal quantification on the variables that are not involved in the substitution given by π_i. Let's take into account the projection $\pi : (x, y) \longrightarrow (x)$, which we will write as $\pi : (x, y) \longrightarrow (z)$ in the spirit of what we said before, and a formula $\psi = \psi(x, y)$ with free variables among x, y, so $[\psi] \in \mathfrak{A}(\mathcal{L})(x, y)$. There must be an adjunction $H_{\mathcal{L}}(\pi) \vdash \forall_\pi(-) = \forall y.(-)[x/z]$, so for any other formula $[\alpha] = [\alpha(z)] \in \mathfrak{A}(\mathcal{L})(z)$ with free variable at most z we need that:

$$[\alpha] \le [\forall y.\psi[x/z]] \Leftrightarrow H_{\mathcal{L}}(\pi)([\alpha]) \le [\psi]$$

but α does not depend on y, hence $H_{\mathcal{L}}(\pi)([\alpha]) = [\alpha]$ and the adjunction is equivalent to the following claim:

$$[\alpha] \le [\forall y.\psi[x/z]] \Leftrightarrow [\alpha] \le [\psi]$$

and this is exactly how it should be. Hence generalizing:

$$H_{\mathcal{L}}(\pi_i) \vdash \forall_{\pi_i}$$

The Beck-Chevalley condition for the functor $\forall$ turns out to rule how to substitute a free variable in an universal quantification. Let's consider the projection $[x/x]$ in **Cont** and a formula $[\psi] = [\psi(x, y)] \in \mathfrak{A}(\mathcal{L})(x, y)$ as before. For every map $[\vec{w}/x] : (w_1, ..., w_n) \longrightarrow (x)$ in **Cont**, the diagram

$$\begin{array}{ccc} (w_1, ..., w_n, y) & \xrightarrow{[\vec{w}/\vec{w}]} & (w_1, ..., w_n) \\ {\scriptstyle [\vec{w}/x, y/y]}\downarrow & & \downarrow{\scriptstyle [\vec{w}/x]} \\ (x, y) & \xrightarrow[{[x/x]}]{\qquad} & (x) \end{array}$$

is a pullback diagram (this was shown some lines above) and we would like the following to hold:

$$[(\forall y.\psi)[x/\vec{w}]] = [\forall y.(\psi[x/\vec{w}, y/y])]$$

so in particular we need the map

$$H_{\mathcal{L}}([\vec{w}/x])\forall_{[x/x]}([\psi]) \leq \forall_{[\vec{w}/\vec{w}]} H_{\mathcal{L}}([\vec{w}/x, y/y])([\psi])$$

to be an isomorphism, which is exactly what the Beck-Chevalley condition claims. As we have done many times so far, we write a projection $(x,y) \longrightarrow (x)$ also as $(x,y) \longrightarrow (z)$. It is easy to see that the Beck-Chevalley condition for such a notation results in the isomorphism:

$$[(\forall y.\psi[x/z])[z/\vec{w}]] = [\forall y.(\psi[x/\vec{w}, y/y])]$$

In the end, for every $(x_{j_1}, ..., x_{j_n}) \in Ob(\mathbf{Cont})$ we define the fibred equality on this object by:

$$\delta_{(x_{j_1},...,x_{j_n})} := x_{j_1} = x_{J+j_1} \wedge ... \wedge x_{j_n} = x_{J+j_n}$$

where $J = max\{j_1, ..., j_n\}$, and that is a formula in $\mathfrak{A}(\mathcal{L})((x_{j_1}, ..., x_{j_n}) \times (x_{j_1}, ..., x_{j_n}))$, thus in $\mathfrak{A}(\mathcal{L})((x_{j_1}, ..., x_{j_n}, x_{J+j_1}, ..., x_{J+j_n})))$. This actually does the trick! Let $(x_{k_1}, ..., x_{k_m}) \in Ob(\mathbf{Cont})$ and let

$$(\pi_1, \pi_2, \pi_2) : (\vec{x_k}, \vec{x_j}) \longrightarrow (\vec{x_k}, \vec{x_j}, \vec{x_{J+j}})$$

be a map in **Cont**. We need to find an adjunction $\exists_{(\pi_1,\pi_2,\pi_3)} \dashv H_{\mathcal{L}}(\pi_1, \pi_2, \pi_3)$, where the left adjoint is:

$$\begin{array}{rccc} \exists_{(\pi_1,\pi_2,\pi_3)}: & \mathfrak{A}(\mathcal{L})(\vec{x_k}, \vec{x_j}) & \longrightarrow & \mathfrak{A}(\mathcal{L})(\mathcal{L})(\vec{x_k}, \vec{x_j}, \vec{x_{J+j}}) \\ & \alpha(\vec{x_k}, \vec{x_j}) & \longmapsto & H_{\mathcal{L}}(1_{\vec{x_k}} \times \pi_1)(\alpha(\vec{x_k}, \vec{x_j})) \wedge H_{\mathcal{L}}(\pi_2, \pi_2)(\alpha(\vec{x_k}, \vec{x_j})) \end{array}$$

so more precisely $H_{\mathcal{L}}(1_{\vec{x_k}} \times \pi_1)(\alpha(\vec{x_k}, \vec{x_j})) \wedge H_{\mathcal{L}}(\pi_2, \pi_2)(\alpha(\vec{x_k}, \vec{x_j})) \equiv \alpha(\vec{x_k}, \vec{x_j}) \wedge (x_{j_1} = x_{J+j_1} \wedge x_{j_2} = x_{J+j_2} \wedge ... \wedge x_{j_n} = x_{J+j_n})$, and the right adjoint is:

$$\begin{array}{rccc} H_{\mathcal{L}}(\pi_1, \pi_2, \pi_3): & \mathfrak{A}(\mathcal{L})(\vec{x_k}, \vec{x_j}, \vec{x_{J+j}}) & \longrightarrow & \mathfrak{A}(\mathcal{L})(\mathcal{L})(\vec{x_k}, \vec{x_j}) \\ & \psi(\vec{x_k}, \vec{x_j}, \vec{x_{J+j}}) & \longmapsto & \psi(\vec{x_k}, \vec{x_j}, \vec{x_j}) \end{array}$$

As a consequence, for any $\alpha(\vec{x_k}, \vec{x_j}) \in \mathfrak{A}(\mathcal{L})(\vec{x_k}, \vec{x_j})$ and any $\psi(\vec{x_k}, \vec{x_j}, \vec{x_{J+j}}) \in \mathfrak{A}(\mathcal{L})(\vec{x_k}, \vec{x_j}, \vec{x_{J+j}})$, we need to have that:

$$\frac{[\alpha(\vec{x_k}, \vec{x_j})] \leq [\psi(\vec{x_k}, \vec{x_j}, \vec{x_j})]}{\overline{[\alpha(\vec{x_k}, \vec{x_j}) \wedge (x_{j_1} = x_{J+j_1} \wedge x_{j_2} = x_{J+j_2} \wedge ... \wedge x_{j_n} = x_{J+j_n})] \leq [\psi(\vec{x_k}, \vec{x_j}, \vec{x_{J+j}})]}}$$

(where the double line is an equivalent notation for the logic equivalence $\Leftrightarrow$). This actually holds and a proof of that can be given by:

$$\cfrac{\cfrac{\begin{array}{c}\alpha(\vec{x_k}, \vec{x_j}) \vdash \psi(\vec{x_k}, \vec{x_j}, \vec{x_j}) \\ \vdots =_1 \\ \alpha(\vec{x_k}, \vec{x_j}), x_{j_1} = x_{J+j_1}, x_{j_2} = x_{J+j_2}, ..., x_{j_n} = x_{J+j_n} \vdash \psi(\vec{x_k}, \vec{x_j}, \vec{x_{J+j}})\end{array}}{\alpha(\vec{x_k}, \vec{x_j}), (x_{j_1} = x_{J+j_1} \wedge x_{j_2} = x_{J+j_2} \wedge ... \wedge x_{j_n} = x_{J+j_n}) \vdash \psi(\vec{x_k}, \vec{x_j}, \vec{x_{J+j}})}\wedge l}{\alpha(\vec{x_k}, \vec{x_j}) \wedge (x_{j_1} = x_{J+j_1} \wedge x_{j_2} = x_{J+j_2} \wedge ... \wedge x_{j_n} = x_{J+j_n}) \vdash \psi(\vec{x_k}, \vec{x_j}, \vec{x_{J+j}})}\wedge l$$

where every rule used in the foregoing proof tree is invertible, so the above logic equivalence is proved.

7.3 Logical quantifiers via adjunctions

We introduced the notion of hyperdoctrine in order to be able to endow the categorical language we have been using with the possibility of expressing quantifications on variables. So far, we have simply introduced new functors, namely $\exists$ and $\forall$, asking some basic rules such as that they respect Beck-Chevalley conditions and Frobenius reciprocity. Thereafter, we considered the particular case of the syntactic hyperdoctrine and we showed that those rules tell something true as regards existential and universal quantifications. However, the whole picture is not yet crystal clear: the heart of the matter is showing that those new theoretical tools actually make possible to speak about quantification as we know it, and it would also be desiderable that adjuntion itself should allow to express quantification in categorical terms (in line with Theorem (6.2) and Theorem (6.3)).

Although it is not clearly stated, the previous section makes conspicuous that this purpose has already been achieved when we formulated the syntactic version of the notion of hyperdoctrine. We would only like to make this all a little bit more evident. In order to do that, we will start from what it is usually required to have in order to work with quantification and we will show that syntactic hyperdoctrines are enough in this respect. Put it another way, we will tackle the problem of a categorical (i.e. by means of adjoint functors) definition of logical quantifiers (we will skip the categorical characterization of equality though), trying to resemble what we did with logical connectives in the previous chapter and, as it is desiderable, we will get to the same exact results of the previous section.

Let $\mathcal{L}$ be a language for the (either intuitionistic or classical) predicate logic and let $\varphi(x_1,...,x_n,y) \in Form_{\mathcal{L}}$ be a proposition with free variables among $x_1,...,x_n,y$, namely $[\varphi(x_1,...,x_n,y)] \in \mathfrak{A}(\mathcal{L})(x_1,...,x_n,y)$. We can consider the existential quantification of that formula $[\exists y.\varphi(x_1,...,x_n,y)]$, as well as the universal one $[\forall y.\varphi(x_1,...,x_n,y)]$, obtaining formulas with free variables among $x_1,...,x_n$, namely formulas in $\mathfrak{A}(\mathcal{L})(x_1,...,x_n)$. In this way, maps

$$\begin{array}{rccc} \exists_y : & \mathfrak{A}(\mathcal{L})(x_1,...,x_n,y) & \longrightarrow & \mathfrak{A}(\mathcal{L})(x_1,...,x_n) \\ & [\varphi(x_1,...,x_n,y)] & \longmapsto & [\exists y.\varphi(x_1,...,x_n,y)] \end{array}$$

and

$$\begin{array}{rccc} \forall_y : & \mathfrak{A}(\mathcal{L})(x_1,...,x_n,y) & \longrightarrow & \mathfrak{A}(\mathcal{L})(x_1,...,x_n) \\ & [\varphi(x_1,...,x_n,y)] & \longmapsto & [\forall y.\varphi(x_1,...,x_n,y)] \end{array}$$

are given. We would also like that quantification behaves properly along substitution of variables. Namely, for any substitution $[\vec{x}/\vec{z}, y/y] : (x_1,...,x_n,y) \longrightarrow (z_1,...,z_m,t)$ we should have that:

$$[(\exists y.\varphi(x_1,...,x_n,y))[\vec{x}/\vec{z},y/y]] = [\exists y.\varphi(z_1,...,z_m,y)] = [\exists y.(\varphi(x_1,...,x_n,y)[\vec{x}/\vec{z},y/y])]$$

and

$$[(\forall y.\varphi(x_1,...,x_n,y))[\vec{x}/\vec{z},y/y]] = [\forall y.\varphi(z_1,...,z_m,y)] = [\forall y.(\varphi(x_1,...,x_n,y)[\vec{x}/\vec{z},y/y])]$$

We find again the Beck-Chevalley conditions for $\exists$ and $\forall$.

Our goal now is mimicking what we did for the connectives of propositional logic: first of all we define functors between Lindenbaum algebras (both in the intuitionistic and in the logical case) that behave like quantifiers do, afterwards we characterize them as the adjoints of some particular functorial structures between Lindenbaum algebras. To begin with, we extend the definition we gave before of Lindenbaum algebra for propositional logic to the predicate case: it suffices to add the possibility to quantify over variables, so we have simply to add variables to the algebra. This is called a Lindenbaum-Tarski algebra.

Definition 7.9. *Let $\mathcal{L}$ be a language for the (either intuitionistic or classical) predicative logic and let $\vec{x} = (x_1,...,x_n)$ be a list of variables. The **Lindenbaum-Tarski algebra** $\mathfrak{A}(\mathcal{L})(\vec{x})$ is the Lindenbaum algebra $\mathfrak{A}(\mathcal{L})$ where formulas have free variables among $(x_1,...,x_n)$, hence quantification is admitted only over variables in $(x_1,...,x_n)$.*

It is straightforward to see that $\mathfrak{A}_i(\mathcal{L})(\vec{x})$ is still a Heyting algebra and $\mathfrak{A}_c(\mathcal{L})(\vec{x})$ is still a Boolean algebra. We can define functors:

$$\begin{array}{llcl}
\exists_y: & \mathfrak{A}_i(\mathcal{L})(x_1,...,x_n,y) & \longrightarrow & \mathfrak{A}_i(\mathcal{L})(x_1,...,x_n) \\
 & [\varphi(x_1,...,x_n,y)] & \longmapsto & [\forall y.\varphi(x_1,...,x_n,y)] \\
 & ([\varphi(x_1,...,x_n,y)] \leq [\psi(x_1,...,x_n,y)]) & \longmapsto & ([\exists y.\varphi(x_1,...,x_n,y)] \leq [\exists y.\psi(x_1,...,x_n,y)])
\end{array}$$

$$\begin{array}{llcl}
\forall_y: & \mathfrak{A}_i(\mathcal{L})(x_1,...,x_n,y) & \longrightarrow & \mathfrak{A}_i(\mathcal{L})(x_1,...,x_n) \\
 & [\varphi(x_1,...,x_n,y)] & \longmapsto & [\exists y.\varphi(x_1,...,x_n,y)] \\
 & ([\varphi(x_1,...,x_n,y)] \leq [\psi(x_1,...,x_n,y)]) & \longmapsto & ([\forall y.\varphi(x_1,...,x_n,y)] \leq [\forall y.\psi(x_1,...,x_n,y)])
\end{array}$$

These functors are well-defined: suppose there is a map $[\varphi(\vec{x},y)] \leq [\psi(\vec{x},y)]$ in $\mathfrak{A}_i(\mathcal{L})(\vec{x},y)$, therefore there is a derivation D of $\varphi(\vec{x},y) \vdash \psi(\vec{x},y)$, hence we can find the following derivations in intuitionistic sequent calculus (and thus they also hold classically)

$$\dfrac{\dfrac{\dfrac{}{\varphi(\vec{x},w) \dashv \psi(\vec{x},w)}}{\varphi(\vec{x},w) \dashv \exists y.\psi(\vec{x},y)}\ \exists r}{\exists y.\varphi(\vec{x},y) \dashv \exists y.\psi(\vec{x},y)}\ \exists l(w \notin FV(\exists y.\varphi(\vec{x},y), \exists y.\psi(\vec{x},y)))$$

$$\dfrac{\dfrac{\dfrac{}{\forall y.\varphi(\vec{x},y), \varphi(\vec{x},w) \dashv \psi(\vec{x},w)}}{\forall y.\varphi(\vec{x},y) \dashv \psi(\vec{x},w)}\ \forall l}{\forall y.\varphi(\vec{x},y) \dashv \forall y.\psi(\vec{x},y)}\ \forall r(w \notin FV(\forall y.\varphi(\vec{x},y), \forall y.\psi(\vec{x},y)))$$

We recall the definition of weakening functor, which we introduced immediately after the definition of syntactic hyperdoctrine. Given a language $\mathcal{L}$ for the predicative logic, let's consider two lists of variables $\vec{x} := (x_1,...,x_n)$ and $(\vec{x},y) = (x_1,...,x_n,y)$ as objects of **Cont**, and let $[\vec{x}/\vec{x}] : (x_1,...,x_n,y) \longrightarrow (x_1,...,x_n)$ be a projection in **Cont**. The weakening functor of the context of the variables $\vec{x}$ with extra variable y

$$H_{\mathcal{L}}([\vec{x}/\vec{x}]) : \mathfrak{A}(\mathcal{L})(\vec{x}) \longrightarrow \mathfrak{A}(\mathcal{L})(\vec{x},y)$$

is the functor image through the syntactic hyperdoctrine $H_{\mathcal{L}}$, sending formulas with free variables in $x_1,...,x_n$ into formulas where we add an extra free variable y.

Theorem 7.1. *Let $\mathcal{L}$ be a language for either the intuitionistic predicative logic IL or the classical predicative logic CL, and let $H_{\mathcal{L}}$ be a functor defined as in Definition (7.8). Let $\vec{x} := (x_1,...,x_n)$, $(\vec{x},y) = (x_1,...,x_n,y) \in Ob(\mathbf{Cont})$ and let $[\vec{x}/\vec{x}] : (x_1,...,x_n,y) \longrightarrow (x_1,...,x_n)$ be a projection in* **Cont**. *Then:*

1. *the existential quantification $\exists_y$ is left adjoint to the weakening functor $H_{\mathcal{L}}([\vec{x}/\vec{x}])$ of the context of the variables $\vec{x}$ with extra variable y:*

$$\exists_y \dashv H_{\mathcal{L}}([\vec{x}/\vec{x}])$$

2. *the universal quantification $\forall_y$ is right adjoint to the weakening functor $H_{\mathcal{L}}([\vec{x}/\vec{x}])$ of the context of the variables $\vec{x}$ with extra variable y:*

$$H_{\mathcal{L}}([\vec{x}/\vec{x}]) \dashv \forall_y$$

Proof. Let's prove the result in the intuitionistic case, so it holds also in the classical one.

Let $[\varphi(\vec{x},y)] \in Ob(\mathfrak{A}_i(\mathcal{L})(\vec{x},y))$. There is a map $[\varphi(\vec{x},y)] \leq [\exists y.\varphi(\vec{x},y)]$ in $\mathfrak{A}_i(\mathcal{L})(\vec{x},y)$ since there is a derivation of $\varphi(\vec{x},y) \vdash \exists y.\varphi(\vec{x},y)$ in intuitionistic sequent calculus:

$$\dfrac{\overset{id-ax}{\varphi(\vec{x},y) \vdash \varphi(\vec{x},y), \exists y.\varphi(\vec{x},y)}}{\varphi(\vec{x},y) \vdash \exists y.\varphi(\vec{x},y)}\ \exists r$$

Let now $[\alpha(\vec{x})] \in Ob(\mathfrak{A}_i(\mathcal{L})(\vec{x}))$ and let $[\varphi(\vec{x},y)] \leq [\alpha(\vec{x})]$ be a map in $\mathfrak{A}_i(\mathcal{L})(\vec{x},y)$, hence $\varphi(\vec{x},y) \vdash \alpha(\vec{x})$. We need to show that there exists a unique map $[\exists y.\varphi(\vec{x},y)] \leq [\alpha(\vec{x})]$, so we have to find a derivation of $\exists y.\varphi(\vec{x},y) \vdash \alpha(\vec{x})$, which can be the following:

$$\dfrac{\dfrac{}{\varphi(\vec{x},y) \vdash \alpha(\vec{x})}}{\exists y.\varphi(\vec{x},y) \vdash \alpha(\vec{x})}\ \exists l (y \notin FV(\exists y.\varphi(\vec{x},y), \alpha(\vec{x})))$$

where a derivation of $\varphi(\vec{x},y) \vdash \alpha(\vec{x})$ is given by assumption and the uniqueness of the map $[\exists y.\varphi(\vec{x},y)] \leq [\alpha(\vec{x})]$ comes from the definition of Lindenbaum algebra.

Let $[\varphi(\vec{x})] \in Ob(\mathfrak{A}_i(\mathcal{L})(\vec{x}))$. There is a map $[\varphi(\vec{x})] \leq [\forall y.\varphi(\vec{x})]$ in $\mathfrak{A}_i(\mathcal{L})(\vec{x})$ since $\forall y.\varphi(\vec{x}) = \varphi(\vec{x})$ ($\varphi(\vec{x})$ does not depend on the variable y), hence there is a derivation of $\varphi(\vec{x},y) \vdash \forall y.\varphi(\vec{x},y)$ in the intuitionistic sequent calculus by identity axiom. Let now $[\alpha(\vec{x},y)] \in Ob(\mathfrak{A}_i(\mathcal{L})(\vec{x},y)$ and let $[\varphi(\vec{x})] \leq [\forall y.\alpha(\vec{x},y)]$ be a map in $\mathfrak{A}_i(\mathcal{L})(\vec{x})$, hence there is derivation tree of $\varphi(\vec{x}) \vdash \forall y.\alpha(\vec{x},y)$ in the intuitionistic sequent calculus. We need to show that there exists a unique map $[\forall y.\varphi(\vec{x},y)] \leq [\alpha(\vec{x})]$, so we have to find a derivation of $\forall y.\varphi(\vec{x},y) \vdash \alpha(\vec{x})$, which is the following[3]:

$$\dfrac{\dfrac{\dfrac{}{\forall y.\varphi(\vec{x},y) \vdash \varphi(\vec{x})} \qquad \dfrac{}{\varphi(\vec{x}) \vdash \forall y.\alpha(\vec{x},y)}}{\forall y.\varphi(\vec{x},y) \vdash \forall y.\alpha(\vec{x},y)}\ comp \qquad \dfrac{}{\forall y.\alpha(\vec{x},y) \vdash \alpha(\vec{x})}}{\forall y.\varphi(\vec{x},y) \vdash \alpha(\vec{x})}\ comp$$

where a derivation of $\varphi(\vec{x}) \vdash \forall y.\alpha(\vec{x},y)$ is given by assumption, derivations of $\forall y.\varphi(\vec{x},y) \vdash \varphi(\vec{x})$ and $\forall y.\alpha(\vec{x},y) \vdash \alpha(\vec{x})$ follow from the fact that both right-hand terms do not depend on y. As a consequence, there is a map $[\exists y.\varphi(\vec{x},y)] \leq [\alpha(\vec{x})]$ in $\mathfrak{A}_i(\mathcal{L})(\vec{x})$, which is unique due to the definition of Lindenbaum algebra. □

7.4 The next step: categorical first order logic

Lawvere's purpose was far more organic than just to express connectives and quantifiers as adjoint functors, that is in a category-theoretic framework. As we already mentioned, his great intuition was to treat categorically first order and higher order logics *on their whole*. Of course, to give a detailed explanation of the matter is beyond the scope of this thesis. Nevertheless, it can be interesting, as well as fair to Lawvere's work, to depict an overall report of the consequent researches and influences of categorical logic. With such a purpose in mind, we will present the Elementary Topos Theory (ETT), which Lawvere formulated together with Tierney and which is an attempt to introduce higher order theories in category-theoretic terms, and we will give a clue of some results that came as an answer to Lawvere's Elementary Theory of the Category of Sets (ETCS).

[3]We make use of the composition rule here; it is obvious that Gentzen Theorem on the elimination of the cut rule allows us to turn this proof tree into one which is cut free.

7.4.1 Elementary Topos Theory

As we have already mentioned, elementary topos theory (ETT) was introduced by Lawvere and Tierney at the beginning of the 1970s. Lawvere's previous work had been an attempt to give to logical concepts a categorical formulation, as well as to capture in category-theoretic terms purely equational theories. The introduction of ETT corresponded to the occasion on which higher order theories, or type theories, were introduced. It arose so an interest in clarifying the relationships between ETT and type theories, as well as in characterizing categorically some intermediate cases, the so called **doctrines**, especially the doctrine of first-order logic.

The concept of **topos** was first introduced by Grothendieck in the early 1960s in the context of algebraic geometry. It was presented as a generalization of the concept of topological space, though Grothendieck himself noticed that a topos inherited many properties of the category of sets and thus could also be treated as a generalization of the notion of category of sets. Nevertheless, Grothendieck and his students did not pay attention to the logical side of the issue. Though social and especially political ideas are rarely though of as being related to mathematical theories, the explanation of the lack of interest for logic during those years can be of this kind. While geometry was considered progressive, logic was meant to be reactionary by many circles. Also, using categorical techniques to solve a problem was sometimes considered off beat and even fascist[4]. This could in part explain why the development of category theory, which was capable more than other branches of mathematics of connecting logic with many different mathematical subjects such as geometry, was so difficult in the initial stages.

However, the pioneering nature of Lawvere overcame these prejudices. When in 1969 his interests in continuum mechanics and Tierney's researches in sheaf theory intersected, the notion of elementary topos arose naturally. In a nutshell, an elementary topos is a CC category with a subobject classifier. More properly, here is the definition as it first appeared in 1969[5].

Definition 7.10. *An **elementary topos** is a category ε such that:*

1. *it has pullbacks;*

2. *it has a terminal object* 1*;*

3. *the functor $X \times - : \varepsilon \longrightarrow \varepsilon$ has a right adjoint $(-)^X : \varepsilon \longrightarrow \varepsilon$, for every $X \in Ob(\varepsilon)$;*

4. *it has a subobject classifier $\Omega \in Ob(\varepsilon)$ and a monic map $\top : 1 \longrightarrow \Omega$ such that for any monic $m : A \longrightarrow X$ there is a unique map $\phi : X \longrightarrow \Omega$ in ε for which the following square is a pullback:*

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, "m"'] & 1 \arrow[d, "\top"] \\ X \arrow[r, "\phi"'] & \Omega \end{tikzcd}$$

   for any $A, X \in Ob(\varepsilon)$ (the map $\phi : X \longrightarrow \Omega$ is usually called characteristic morphism of A and is usually denoted by ϕ_A).

Lawvere guessed the logical structure of a topos right from the start. It can actually be showed that all propositional operations are definable in an alementary topos ε as maps of the subobject

[4] The political influence during those year was pervasive: it suffices to think that for instance set theory was said to be bourgeois, since it was founded on the relationship of belonging.

[5] We underline the fact that the definition can be given entirely in terms of adjunction.

classifier Ω:

$$\begin{aligned} \top &: 1 \longrightarrow \Omega \\ \neg &: \Omega \longrightarrow \Omega \\ \bot \equiv \neg \circ \top &: 1 \longrightarrow \Omega \\ \wedge &: \Omega \times \Omega \longrightarrow \Omega \\ \vee &: \Omega \times \Omega \longrightarrow \Omega \\ \rightarrow &: \Omega \times \Omega \longrightarrow \Omega \end{aligned}$$

Furthermore, for any map $f : X \longrightarrow Y$ in ε there is are map $\exists_f : \Omega^X \longrightarrow \Omega^Y$ and $\forall_f : \Omega^X \longrightarrow \Omega^Y$ which can be given as adjoints to given functors. Consequently, quantifiers are definable in any elementary topos, too.

It turns out that the logic of an arbitrary topos ε is in general intuitionistic, though there exist Boolean topoi too: for instance it is sufficient that the negation operator $\neg$ satisfies $\neg \circ \neg = 1_\Omega$, or equivalently that $\top$ and $\bot$ induce an isomorphism $1+1 \simeq \Omega$ so the subobject classifier is two-valued. In 1971 Diaconescu found out that a sufficient condition for a topos to be Boolean is that it satisfies the axiom of choice (AC)[6]. Later it was also showed that any topos satisfying AC also satisfies the internal axiom of choice (IAC)[7] and that any topos satisfying IAC is necessarily Boolean.

A morphism of topoi is naturally defined as a functor preserving the structure. Lawvere and Tierney called **logical morphism** such a functor, but they preferred to use instead **geometric morphisms** in the development of their theory. These geometric morphisms had already been defined and heavily used by Grothendieck and his students, also they guided Lawvere and Tierney up to the definition of **Grothendieck topology** (also known as **Lawvere-Tierney topology**), which made possible to put the geometry at the forefront of their theory.

What is relevant, though, is that ETT was at that point ready to let Lawvere and Tierney formalize Cohen's independence proof in categorical terms: in 1970 Lawvere translated Cohen's proof into the language of topos theory and two years later Tierney presented the full proof of this translation ([Tie72]).

As it should be clear from what we have just reported, categorical logic raised, starting from its early origins, a huge amount of foundational questions, presenting itself explicitly as a reversal of the traditional presentation of mathematical concepts and theories. Whereas in most set-theoretical accounts the approach to mathematics is clearly inherited by Russell's philosophical and logical visions and is **atomistic** (or **bottom-up**, according to Awodey in *An answer to Hellman's question: 'Does category theory provide a framework for mathematical structuralism?'*, Philosophia Mathematica, 12, 2004), the approach to mathematics proposed by category theory is on the contrary **algebraic** (or **top-down** according to Awodey in *An answer to Hellman's question: 'Does category theory provide a framework for mathematical structuralism?'*, Philosophia Mathematica, 12, 2004). According to logical atomism, all truths are ultimately dependent upon a layer of atomic facts, which consist either of a simple particular exhibiting a quality, or multiple simple particulars standing in a relation. According to the algebraic vision of mathematics and especially logic, the truth of the propositional atoms is subsequent to some algebraic structures they belong to.

Having said that, the foundational opinions and beliefs inside the categorical community differentiate one from each other for specific important aspects. There are some foundational elements, though, on which all categorical logicians of the period agreed. First, all categorical logicians postulated that, by adopting a top-down approach to the analysis of mathematical concepts, structures shared by different mathematical systems should be described in terms of "maps" (here not meant explicitly as maps in a category) between them. In the case of Lawvere's work, maps are meant as adjoint functors between different categories that formalize some mathematical structures. Second, it was a common belief in the categorical community that mathematics did not require a

[6]A topos satisfies AC if every epimorphism $p : X \longrightarrow Y$ hass a section $s : Y \longrightarrow X$, that is $p \circ s = 1_Y$.

[7]A topos ε satisfies IAC if for every epimorphism $p : X \longrightarrow Y$ and for any object $Z \in Ob(\varepsilon)$, the map $X^Z \longrightarrow Y^Z$ is also epic.

unique and absolute foundation, as for instance set theory had been meant up to those days; also, they thought that certain frameworks even logically weaker than ZF could be as well satisfactory. Of course, categorical logic was believed to provide the most handy tools for the analysis of the logical structure of many mathematical disciplines. Third, and as consequence to what has just been said, category theorists and categorical logicians agreed that it was not necessary to assume that mathematics was about sets: sets are indeed not constitutive of the structure of categories themselves, since there are many kinds of categories that do not have a set structure.

Starting from these elements as a common base, every categorical logician elaborated his own foundational theory. Among the many, five positions in particular were the most important in the development of the matter and were presented by: Lawvere, Lambek, MacLane, Bell and Makkai.

In reality, Lawvere's opinion has changed through the years, even though a constant point is the conviction that category theory provides the proper setting for the study of foundations. However, what Lawvere meant for mathematical foundations should be made clear at the outset, since his philosophical and mathematical considerations are manifold and different depending on the years they are considered. Despite that, although Lawvere himself is aware of facing foundational problems differently at different times, he cares about clarifying that his purpose has always been steadfast: to produce a context in which mathematical domains may be characterized categorically (and more precisely, by means of naturally arising adjoint functors to canonical given functors), so that a top-down approach to mathematics may be undertaken. Actually, his position can be said to be deeply historical and dialectical, far more than top-down. Using Lawvere's words:

> "[a] foundation makes explicit the essential features, ingredients, and operations of a science as well as its origins and general laws of development. The purpose of making these explicit is to provide a guide to the learning, use, and further development of the science. A 'pure' foundation that forgets this purpose and pursues a speculative 'foundation' for its own sake is clearly a nonfoundation." ([LR03], pg 235)

This, together with the idea that there is no such thing as *the* foundation for mathematics, implies that, in Lawvere's opinion, foundations should not be perscriptive about what consitutes mathematics, but rather descriptive about its *origins* and *essential features*.

Lambek's position is totally different from Lawvere's. He too is clearly concerned with the history of mathematics, but he keeps this interest separated from a his deeply elaborated philosophical work ([LS80]), in which he focused on the analysis of how the standard philosophical positions in the foundations (logicism, intuitionism, formalism, platonism) might be inserted in a topos-theoretical approach to mathematics.

A different view held MacLane, who at first, after founding category theory, did not see it as more than a handy and useful language. Under the influence of Lawvere's ideas, he reconsidered foundational issues and raised many questions as to set-theoretical foundations for category theory. Basically, he recalled Lawvere's ETCS in a topos-theoretical setting, asserting that a well-pointed topos with choice and a natural-number object could offer an alternative to ZFC, and so that a different foundation for mathematics was possible. Nevertheless, MacLane's complete picture of his foundational ideas can be found in his book *Mathematics Form and Function* ([Mac86]): in a nutshell, mathematics is presented as arising from a formal network based on ideas and concepts that evolve through time according to their function and consequently there cannot be any preference for a set-theoretic or category-theoretic perspective as regards the foundations of mathematics, which is seen as form and function.

Bell, like Lambek, had a profound interest on the historical influence on foundational issues and, like Lawvere, adopted a distinctly dialectical attitude towards the subject. Even though in the early 1980s he had totally denied the possibility for category theory to be taken as a foundational framework, some years later he began to acknowledge category theory to play a foundational role in mathematics. More precisely, he advanced the idea that toposes and the associated higher-order intuitionistic type theories provided a network in which one could analyze mathematical concepts. Differently from Lambek, though, his approach is pluralist and top-down, since Bell did not argue in favor of a specific kind of toposes. In his words:

> "the topos-theoretical viewpoint suggests that the absolute universe of sets be replaced by a plurality of 'toposes of discourse', each of which may be regarded as a possible 'world' in which mathematical activity may (figuratively) take place. The mathematical activity that takes place within such 'worlds' is codified within local set theories; it seems appropriately, therefore, to call this codification local mathematics, to constrast it with the absolute universe of sets. Constructive probability of a mathematical assertion now means that it is invariant, i.e. valid in every local mathematics." ([Bel88], pg 245)

Makkai's contributions were not only technical, but also of philosophical relevance. From a technical point of view, he agreed that a topos-theoretical perspective could not provide an adequate foundation for category theory, therefore he searched for an appropriate metatheory for category theory trying, following in Lawvere's footsteps, to provide a metatheoretic description of the category of categories. Philosophically, he looked into the connections between these issues and mathematical structuralism, which was, according to him, the belief that conceptual world consisted on structures. His crucial idea was that the identity relation in formal languages was not given a priori by first-order axioms, but was derived from within a certain context. Practically speaking, only when the context for talking about share structures has been fixed, can a criterion of identity for objects of that structure be given directly by the context itself (this is in fact a structurally interpreted context principle).

7.4.2 First order logic in a new mindset: Joyal and Reyes' strategy

As we have already mentioned, Lawvere's first attempt at categorizing set theory was the Elementary Theory of the Category of Sets (ETCS). However, this did not give brilliant results: strictly speaking, it was nothing but a categorical transcription of the axioms of set theory and did not give any new insight into the subject. Lawvere's interests flared up again after the discovery of the importance of the concept of topos in logic. His great idea was to replace the category of sets by an arbitrary topos, in order to develop mathematics, in particular analysis. The axiom of topos theory seemed to be very much set-like, thus the biggest initial efforts were concerned with the clarification of the relationships between topos and set theory.

As regards first-order logic, its development is difficult to report accurately. The most authoritative references are for sure [Law75a], [Law75b] and [Law76], though there were many others all-important contributions by Reyes, Lambek, Makkai, Joyal and Bénabou. In case they were published, these contributions were surprisingly difficult to find and examine, but Bénabou and Joyal never even wanted to publish their results. In Marquis and Reyes' opinion ([MR11]), historical events might have had an influence: both in France and in Montreal, which where the places at the core of the development of the subject, social, political and cultural situation was very tense (recall for instance the *October Crisis* in Québec in 1970 or *May 68* in France). Bénabou and Joyal gave only talks in conferences, colloquia and seminars, but no specific details about their work were available. Two important events in the development of categorical logic and first-order logic theories were two meetings in April 1973 in Montreal, where Dana Schlomiuk had invited many category theorists to a series of conferences and seminars, and in the summer of 1974, when Shuichi Takahashi had organized the *Séminaire de mathématiques supérieures*. There were no official records of these meetings, though.

The first published record of all those researches came out only in 1973, when Hugo Volger, one of Lawvere's students, presented a categorical version of a first-order logic and of the completeness theorem by following the guidelines of his supervisor. Nevertheless, Volger's proposal was not received favourably and the community did not adopt his results: many notions he presented could not be extended to the non-Boolean cases and the whole proposal seemed to most category theorists too contrived and ad hoc. More appreciated answers came from another front, namely from the researches of Joyal and Reyes, who were working in Montreal school and were looking at categorical logic from another point of view. Their idea was to start from a logical theory and then construct a corresponding category: in this way a hierarchy of logical theories is introduced hand-in-hand with a hierarchy of categories. Thanks to this approach, Joyal and Reyes succeeded in finding connections with elementary topoi and Joyal was capable of giving a uniform categorical treatment of the various completeness theorems. The whole work of the two mathematicians can be found in a publication by Reyes in 1974 ([Rey74]), even though there can be also found eight thesis, Master's and PhD's, touching this subject and written at the University of Montreal between 1973 and 1977.

Let's look at Joyal and Reyes' strategy attentively: given a first-order theory $\mathbb{T}$, they wanted to construct a corresponding small category $\mathcal{C}_{\mathbb{T}}$ and then characterize in an abstract manner the type of the category thus obtained. Except for its successful results, as we are going to see, there were also philosophical reasons indicating that this approach was the "right" one: Joyal and Reyes retraced the canonical Lindenbaum-Tarski construction in categorical terms in such a way that it yielded a category. This gave evidence of the fact that categorical logic provided a natural setting for the algebraic treatment of logic. Besides, as Jean Dionne suggested in the introduction of his thesis, categorical logicians believed that substituting the theory $\mathbb{T}$ with the category $\mathcal{C}_{\mathbb{T}}$ would make simpler and more complete the object of study, since categories allowed for more relations between concepts than the canonical set-theoretic inclusion. Thus in the early 1970s, categorical methods were believed to be more objective and invariant than logical methods.

> "Our method is therefore general enough to apply to a large variety of languages and we can conclude that it constitutes the true link between logic in the traditional sense and categorical logic: with the advantage that we can now work without using axioms, rules of inference, formal demonstrations and the rest. We can even forget about variables, which were always inconvenient. Moving to categories, we can do logic in an "abstract" fashion in terms of objects, arrows, functors and diagrams, which is, among all advantages mentioned in this work, not the least, since the logician is naturally more interested by concepts than by indices to give to variables." ([Dio73], translation by Marquis and Reyes)

The first step in Joyal and Reyes' construction is the definition of a first-order primitive language $\mathcal{L}_0$ (which contains only variables, functional and relational symbols, and the true formula $\top$), followed by the presentation of its axioms and rules, which in fact are those of equational logic. The first extension of $\mathcal{L}_0$, namely $\mathcal{L}_1$ or $\mathcal{L}_{reg}$, is called **regular logic** and contains the conjunction connective $\wedge$ and the existential quantifier $\exists$. The following step is considering also the disjunction connective $\vee$ and the false formula $\bot$, thus obtaining the next extension of the language, namely $\mathcal{L}_2$ or $\mathcal{L}_{coh}$, known as **coherent logic**. The last two levels are the intuitionistic and the classical ones. In order to obtain **intuitionistic logic** $\mathcal{L}_3$ or $\mathcal{L}_{int}$, it suffices to extend $\mathcal{L}_{coh}$ with the implication connective $\to$, the universal quantifier $\forall$ and their intuitionistic rules. Finally, by adding the negative connective $\neg$ and the disjuntion rule $\top \vdash \phi \vee \neg\phi$, **classical logic** $\mathcal{L}_4$ or $\mathcal{L}_{bool}$ is obtained.

$$\begin{tikzcd} & & & \mathcal{L}_{int} \\ \mathcal{L}_0 \arrow[r, dash] & \mathcal{L}_{reg} \arrow[r, dash] & \mathcal{L}_{coh} \arrow[ur, dash] \arrow[dr, dash] & \\ & & & \mathcal{L}_{bool} \end{tikzcd}$$

Let $\mathbb{T}$ be a given theory. Then we can construct the corresponding category $\mathcal{C}_{\mathbb{T}}$ in the following manner. First of all, we define the equivalence relation $\sim$:

$$\varphi(\vec{x}) \sim \psi(\vec{x}) \Leftrightarrow (\varphi(x) \vdash_{\mathbb{T}} \psi(\vec{x}) \text{ and } \psi(\vec{x}) \vdash_{\mathbb{T}} \varphi(\vec{x}))$$

The equivalence class of a formula $\varphi(\vec{x})$ is denoted by $[\varphi(\vec{x})]$, is called a **formal set** and constitutes an object of the category $\mathcal{C}_{\mathbb{T}}$. Maps between formal sets in $\mathcal{C}_{\mathbb{T}}$ are called **formal functions** and are defined as equivalent classes of formulas that are provably equivalent functional relations of $\mathbb{T}$.

Specifically, given $[\varphi(\vec{x})], [\psi(\vec{y})] \in Ob(\mathcal{C}_{\mathbb{T}})$ such that the contexts $\vec{x}$ and $\vec{y}$ are disjoint, a formula $\rho(\vec{x}, \vec{y})$ defines a formal function from $[\varphi(\vec{x})]$ to $[\psi(\vec{y})]$, denoted $\langle \vec{x} \longmapsto \vec{y} : \rho \rangle : [\varphi(\vec{x})] \longrightarrow [\psi(\vec{y})]$, if:

1. $\rho(\vec{x}, \vec{y}) \vdash_{\mathbb{T}} \varphi(\vec{x}) \wedge \psi(\vec{y})$;
2. $\rho(\vec{x}, \vec{y}) \wedge \rho(\vec{x}, \vec{z}) \vdash_{\mathbb{T}} \vec{y} = \vec{z}$;
3. $\varphi(\vec{x}) \vdash_{\mathbb{T}} \exists \vec{y}.\rho(\vec{x}, \vec{y})$;

and two formal functions from $[\varphi(\vec{x})]$ to $[\psi(\vec{y})]$ are identified whenever:

$$\rho(\vec{x}, \vec{y}) \vdash_{\mathbb{T}} \sigma(\vec{x}, \vec{y}) \text{ and } \sigma(\vec{x}, \vec{y}) \vdash_{\mathbb{T}} \rho(\vec{x}, \vec{y})$$

The moral is that for each theory $\mathbb{T}$ there is a corresponding small category $\mathcal{C}_{\mathbb{T}}$, as the following table shows.

If $\mathbb{T}$ is in	**then $\mathcal{C}_{\mathbb{T}}$ is a**
$\mathcal{L}_{reg}$	regular category
$\mathcal{L}_{coh}$	coherent category
$\mathcal{L}_{int}$	Heyting category
$\mathcal{L}_{reg}$	Boolean category

At this point, as we have already said, the plan was to describe the properties of the categories $\mathcal{C}_{\mathbb{T}}$ corresponding to theories $\mathbb{T}$, since it was assumed that replacing $\mathbb{T}$ with $\mathcal{C}_{\mathbb{T}}$ was not only mathematically fruitful but also philosophically motivated.

Definition 7.11. *A category $\mathcal{C}$ is said to be a **regular category** if it satisfies the following properties:*

1. *it has all finite limits;*
2. *coequalizers of kernel pairs exist*[8]*;*
3. *regular epimorphisms are stable under pullback*[9]*.*

For example, the category **Set** of sets is regular and in reality so is any elementary topos. It is possible to define **regular functors** between regular categories as functors preserving finite limits and coequalizers of kernel pairs. Then we say **Reg** to be the category where objects are (small) regular categories and maps are regular functors between them.

Definition 7.12. *A category $\mathcal{C}$ is said to be a **coherent category** if it satisfies the following properties:*

1. *it is a regular category;*
2. *for each object $X \in Ob(\mathcal{C})$, $Sub(X)$ has finite joins;*
3. *finite joins are stable under pullbacks (equivalently, for every $f : X \longrightarrow X$ in $\mathcal{C}$, the inverse image map $f^* : Sub(Y) \longrightarrow Sub(X)$ preserves finite joins).*

A functor between regular categories is said to be a **coherent functor** if it is regular and it preserves finite joins. Again, we can consider the category **Coh** whose objects are (small) coherent categories and whose maps are coherent functors between them.

[8]A **kernel pair** of a map $f : X \longleftrightarrow Y$ in the category $\mathcal{C}$ is the pullback of a pair of equal maps $Y \xrightarrow{f} Y \xleftarrow{f} X$.
[9]This means that if $f : X \longrightarrow Y$ is a regular epimorphism and $g : Z \longrightarrow Y$ is a map in $\mathcal{C}$, then the map $g' : X \times_Y Z \longrightarrow X$ obtained by pulling back f along g is a regular epimorphism.

Definition 7.13. *A category $\mathcal{C}$ is said to be a* ***Heyting category*** *if it satisfies the following properties:*

1. *it is a regular category;*
2. *for every map $f : X \longrightarrow Y$, the inverse image map $f^* : Sub(Y) \longrightarrow Sub(X)$ has a right adjoint $\forall_f : Sub(X) \longrightarrow Sub(Y)$;*
3. *for every object $X \in Ob(\mathcal{C})$, $Sub(X)$ is a lattice with smallest element.*

A **Heyting functor** between Heyting categories is a regular functor preserving $\forall_f$ and the lattice operations. Similarly as before, it is possible to define the category **Heyt** of Heyting categories and Heyting functors between them.

Definition 7.14. *A category $\mathcal{C}$ is said to be a* ***Boolean category*** *if it satisfies the following properties:*

1. *it is a Heyting category;*
2. *for every object $X \in Ob(\mathcal{C})$, $Sub(X)$ is a Boolean algebra.*

A **Boolean functor** is a functor between Boolean categories preserving all the lattice operations and the category **Bool** has Boolean categories as objects and Boolean functors between them. Quite apparently, there are forgetful functors (actually 2-functors):

$$\mathbf{Bool} \longrightarrow \mathbf{Heyt} \longrightarrow \mathbf{Coh} \longrightarrow \mathbf{Reg}$$

Some questions arose immediately. First of all, it could be interesting to look into possible adjoints to the foregoing functors. Besides, there should be some relationship between these categories and topoi, which could be useful to understand fully the whole matter. Reyes and Joyal answered to those questions in their publication of 1974. The connections with topoi made it possible to formalize a dictionary between algebraic geometry and logic, which was ready to be used to find out new and useful mathematical insights.

Nevertheless, something was still missing, namely a way to construct a theory $\mathbb{T}_{\mathcal{C}}$ given a category $\mathcal{C}$ in such a way that there is a canonical interpretation of $\mathbb{T}_{\mathcal{C}}$ in $\mathcal{C}$, once that the notion of interpretation of a theory in a category has been clarified. Now, we do not think that getting bogged down into too specific detail can be somehow useful for our target, which is simply an overall view of the historic events of the development of categorical logic. It suffices to say that, given a category $\mathcal{C}$ and a first-order language $\mathcal{L}_i$ $(i = 0, ..., 4)$, a proper definition of $\mathcal{C}$-interpretation makes it possible the formalization of the notion of model of a sequent and of a theory $\mathbb{T}$. Also, we can define the internal language $\mathcal{L}_{\mathcal{C}}$ of the category $\mathcal{C}$, which in fact is nothing less that $\mathcal{C}$ itself, and the canonical interpretation of $\mathcal{L}_{\mathcal{C}}$ in $\mathcal{C}$, which is nothing less that the identity interpretation. The theory $\mathbb{T}_{\mathcal{C}}$ corresponding to the category $\mathcal{C}$ is then the collection of all sequents of the internal language that are verified in $\mathcal{C}$ by the canonical interpretation. The point of this kind of tautologous game is that one can prove properties of the category $\mathcal{C}$ using the appropriate deductive system related to the theory $\mathbb{T}_{\mathcal{C}}$. More properly, we should say that the internal language $\mathcal{L}_{\mathcal{C}}$ of the category $\mathcal{C}$ is a special case of the primitive language $\mathcal{L}_0$, and we better indicate the theory $\mathbb{T}_{\mathcal{C}}$ by $\mathbb{T}_0(\mathcal{C})$. If $\mathcal{C}$ is a regular category, then we obtain a theory $\mathbb{T}_1(\mathcal{C})$ by adding suitable axioms to $\mathbb{T}_0(\mathcal{C})$. Similarly, we can associate a specific theory $\mathbb{T}_i(\mathcal{C})$ to coherent, Heyting and Boolean categories.

If $\mathcal{C}$ is a	**then $\mathcal{L}_{\mathcal{C}}$ is**	**and $\mathbb{T}_{\mathcal{C}}$ is**
category	$\mathcal{L}_0(\mathcal{C})$	$\mathbb{T}_0(\mathcal{C})$
regular category	$\mathcal{L}_1(\mathcal{C})$	$\mathbb{T}_1(\mathcal{C})$
coherent category	$\mathcal{L}_2(\mathcal{C})$	$\mathbb{T}_2(\mathcal{C})$
Heyting category	$\mathcal{L}_3(\mathcal{C})$	$\mathbb{T}_3(\mathcal{C})$
Boolean category	$\mathcal{L}_4(\mathcal{C})$	$\mathbb{T}_4(\mathcal{C})$

It is straightforward to prove a soundness theorem for the various logical systems in the table. Much more intricate turns out to be the proof of the completeness theorems, and for this reason we do not dwell on the subject, though we should underline that the researches in that respect brought new tools and brilliant results in categorical logic and some other mathematical areas.

What counts now is the fact that, in a sense, the border between logical theories as categories and categories as logical theories seems to vanish. This is indeed the real terrific strength of categorical logic, or more generally of topos theory, as Reyes noticed:

> ”The goal of topos theory is to develop a language and an efficient method for the study of concepts of local character (as well as constructions on such concepts) that are found in different branches of mathematics: topology, geometry, analytic geometry,... To the geometric aspect (or topological), which is the dominant aspect, another is dialectically opposed: the logical aspect.” ([Rey78], translation by Reyes and Marquis)

Chapter 8

Soundness and completeness

The Theorem of soundness and completeness for some kind of logic is a well-known result of standard mathematical logic. It states that a logical inference is provable if and only if it is true[1], so it establishes a close connection between logical provability and semantical truth.
In this chapter we will focus only on the intuitionistic case, as the classical case arises as a natural generalization. Our aim now is to give a proof of the Theorem of soundness and completeness for predicate logic. Previously, the notion of interpretation needs to be presented: in fact hyperdoctrines themselves will serve as interpretations and so also as models for logical systems. Afterwards, we will extend the result to predicate logic with equality. In order to be able to produce models and countermodels for formulas in a clear and easy way, we will previously give a particular categorical-theoretic version of the notion of interpretation and we will exemplify its use in the last section of the chapter.
On the top of all that, we have to underline that the proof of the Theorem of soundness and completeness via categorical logic and hyperdoctrines, both in the intuitionistic and in the classical case, turns out to be totally constructive, instead it is known that the proof of the completeness Theorem via Boolean algebraic models requires Zorn's Lemma, hence not being constructive.

8.1 Term and formula interpretation under context

With the notation $t_{(x_1,..,x_n)}$ or $\varphi_{(x_1,...,x_n)}$ we mean that either the term t or the formula φ has free variables among $x_1,...,x_n$.

Definition 8.1. *Let $\mathcal{L}$ be a language for the (intuitionistic or classical) predicative logic with $(c_i)_{i\in I}$ in it, $(f_i)_{i\in I}$ be terms and $(P_j)_{j\in J}$ be atomic predicates. Let $\mathfrak{D}$ be a hyperdoctrine for the classical or the intuitionistic logic from a category $\mathcal{C}$, which has all the finite products. Given $C \in Ob(\mathcal{C})$, let be given an assignation:*

- $(c_i)^{\mathcal{I}} : 1 \longrightarrow C$;
- $(f_i)^{\mathcal{I}} : C^{n_i} \longrightarrow C$ *where n_i is the arity of f_i;*
- $(P_j)^{\mathcal{I}} \in \mathfrak{D}(C^{n_j})$ *where n_j is the arity of P_j.*

We define the ***interpretation of terms and formulas based on the hyperdoctrine*** $\mathfrak{D}$ *by induction on the construction of terms and formulas.*

The ***term interpretation*** *of a term $t_{(x_1,...,x_n)}$, with free variables among $x_1,...,x_n$, is a map*

$$t^{\mathcal{I}} : C^n \longrightarrow C$$

in $\mathcal{C}$, defined by induction as follows:

- $\mathcal{I}(x_{i(x_1,...,x_n)}) := \pi_i : C^n \longrightarrow C$ *with π_i projection*

- $\mathcal{I}(f_i(t_1,...,t_m)_{(x_1,...,x_n)}) \coloneqq (f_i)^{\mathcal{I}}(\mathcal{I}(t_1),...,\mathcal{I}(t_m))$

*The **formula interpretation** of a formula $\varphi_{(x_1,...,x_n)}$, with free variables among $x_1,...,x_n$, is the assignation*

$$\mathcal{I}(\varphi_{(x_1,...,x_n)}) \in \mathfrak{D}(C^n)$$

defined by induction on the complexity of the formula $\varphi(x_1,...,x_n)$ as follows:

- $\mathcal{I}(P_j(t_1,...,t_{m_j})_{(x_1,...,x_n)}) \coloneqq \mathfrak{D}(t_1^{\mathcal{I}} \times ... \times t_{m_j}^{\mathcal{I}})(P_j^{\mathcal{I}})$
- $\mathcal{I}(\neg\psi)_{(x_1,...,x_n)} \coloneqq \neg\mathcal{I}(\psi)_{(x_1,...,x_n)} \in \mathfrak{D}(C^n)$
- $\mathcal{I}(\psi \wedge \chi)_{(x_1,...,x_n)} \coloneqq \mathcal{I}(\psi_{(x_1,...,x_n)}) \wedge \mathcal{I}(\chi_{(x_1,...,x_n)}) \in \mathfrak{D}(C^n)$
- $\mathcal{I}(\psi \vee \chi)_{(x_1,...,x_n)} \coloneqq \mathcal{I}(\psi_{(x_1,...,x_n)}) \vee \mathcal{I}(\chi_{(x_1,...,x_n)}) \in \mathfrak{D}(C^n)$
- $\mathcal{I}(\psi \to \chi)_{(x_1,...,x_n)} \coloneqq \mathcal{I}(\psi_{(x_1,...,x_n)}) \to \mathcal{I}(\chi_{(x_1,...,x_n)}) \in \mathfrak{D}(C^n)$
- $\mathcal{I}(\forall x_{n+1}.\psi(x_{n+1}))_{(x_1,...,x_n)} \coloneqq \forall_{\pi_{n+1}}\mathcal{I}(\psi_{(x_1,...,x_n)}) \in \mathfrak{D}(C^n)$
- $\mathcal{I}(\exists x_{n+1}.\psi(x_{n+1}))_{(x_1,...,x_n)} \coloneqq \exists_{\pi_{n+1}}\mathcal{I}(\psi_{(x_1,...,x_n)}) \in \mathfrak{D}(C^n)$

Before proceeding with the proofs of soundness and completeness, we need to verify the validity of some crucial lemmas, such as the weakening and the substitution lemma, as well as checking that the identical assignation gives rise to the identical interpretation (so our definition of interpretation works exactly as it is supposed to).

In the interest of brevity, we consider only the intuitionistic case, as the classical case is a simple generalization of it.

Lemma 8.1 (weakening)**.** *Let $\mathcal{L}$ be a language for the intuitionistic predicative logic and let $\mathfrak{D}$ be a hyperdoctrine for the intuitionistic logic from a category $\mathcal{C}$. Let $t_{(x_1,...,x_n)}$ be a term and $\varphi_{(x_1,...,x_n)} \in Form_{\mathcal{L}}$ a formula, with free variables among $x_1,...x_n$ and $y \neq x_i$ $(i = 1,...,n)$. Then:*

(i) $\mathcal{I}(t)_{(x_1,...x_j,y,x_{j+1},...,x_n)} = t^{\mathcal{I}}_{(x_1,...x_j,x_{j+1},...,x_n)} \circ (\pi_1 \times ... \times \pi_j \times \pi_{j+1} \times ... \times \pi_n)$

(ii) $\mathcal{I}(\varphi)_{(x_1,...x_j,y,x_{j+1},...,x_n)} = D(\pi_1 \times ... \times \pi_j \times \pi_{j+1} \times ... \times \pi_n)(\mathcal{I}(\varphi)_{(x_1,...x_j,x_{j+1},...,x_n)})$

Proof. In the interests of brevity, we will write $(\vec{x}) \equiv (x_1,...,x_j,x_{j+1},...,x_n)$, $(-,y,-) \equiv (x_1,...,x_j,y,x_{j+1},...,x_n)$ and $(\vec{\pi}) \equiv (\pi_1 \times ... \times \pi_n)$.

(i) Let's proceed by induction on the complexity of the term t.

- If $t \equiv x_i$, then $t^{\mathcal{I}}_{(\vec{x})} = \mathcal{I}(x_i)_{(\vec{x})} = \pi_i^n : C^n \longrightarrow C$ and $t^{\mathcal{I}}_{(-,y-)} = \mathcal{I}(x_i)_{(-,y,-)} = \pi_i^{n+1} : C^{n+1} \longrightarrow C$. Thus:
$$t^{\mathcal{I}}_{(-,y-)} = \pi_i^{n+1} = \pi_i^n \circ (\vec{\pi}) = t^{\mathcal{I}}_{(\vec{x})} \circ (\vec{\pi})$$
- If $t \equiv f_i(t_1,...,t_m)_{(\vec{x})})$, then $\mathcal{I}(f_i(t_1,...,t_m)_{(\vec{x})}) = (f_i)^{\mathcal{I}}(\mathcal{I}(t_1)_{(\vec{x})},...,\mathcal{I}(t_m)_{(\vec{x})})$ and similarly $\mathcal{I}(f_i(t_1,...,t_m)_{(-,y,-)}) = (f_i)^{\mathcal{I}}(\mathcal{I}(t_1)_{(-,y,-)},...,\mathcal{I}(t_m)_{(-,y,-)})$. Thus:
$$\begin{aligned}
&\mathcal{I}(f_i(t_1,...,t_m)_{(-,y,-)}) = \\
&= (f_i)^{\mathcal{I}}(\mathcal{I}(t_1)_{(-,y,-)},...,\mathcal{I}(t_m)_{(-,y,-)}) = \\
&= (f_i)^{\mathcal{I}}(\mathcal{I}(t_1)_{(\vec{x})} \circ (\vec{\pi}),...,\mathcal{I}(t_m)_{(\vec{x})} \circ (\vec{\pi})) = \\
&= (f_i)^{\mathcal{I}}((\mathcal{I}(t_1)_{(\vec{x})},...,\mathcal{I}(t_m)_{(\vec{x})}) \circ (\vec{\pi})) = \\
&= (f_i)^{\mathcal{I}} \circ (\mathcal{I}(t_1)_{(\vec{x})},...,\mathcal{I}(t_m)_{(\vec{x})}) \circ (\vec{\pi}) = \\
&= \mathcal{I}(f_i(t_1,...,t_m))_{(\vec{x})} \circ (\vec{\pi}) = \\
&= f_{i}{}^{\mathcal{I}}_{(\vec{x})} \circ (\vec{\pi})
\end{aligned}$$

(ii) Let's proceed by induction of the complexity of the formula $\varphi \in Form_{\mathcal{L}}$.

- If $\varphi \equiv P_j(t_1, ..., t_{m_j})$, then $\mathcal{I}(P_j(t_1, ..., t_{m_j})_{(\vec{x})}) = \mathfrak{D}(t^{\mathcal{I}}_{1\,(\vec{x})} \times ... \times t_{m_j}{}^{\mathcal{I}}_{(\vec{x})})(P^{\mathcal{I}}_j)$ and $\mathcal{I}(P_j(t_1, ..., t_{m_j})_{(-,y,-)}) = \mathfrak{D}(t^{\mathcal{I}}_{1\,(-,y,-)} \times ... \times t^{\mathcal{I}}_{m_j\,(-,y,-)})(P^{\mathcal{I}}_j)$. Thus:

$$\begin{gathered}
\mathcal{I}(P_j(t_1, ..., t_{m_j})_{(-,y,-)}) = \\
= \mathfrak{D}(t^{\mathcal{I}}_{1\,(-,y,-)} \times ... \times t^{\mathcal{I}}_{m_j\,(-,y,-)})(P^{\mathcal{I}}_j) = \\
= \mathfrak{D}((t^{\mathcal{I}}_{1\,(\vec{x})} \circ (\vec{\pi})) \times ... \times (t^{\mathcal{I}}_{m_j\,(\vec{x})} \circ (\vec{\pi})))(P^{\mathcal{I}}_j) = \\
= \mathfrak{D}((t^{\mathcal{I}}_{1\,(\vec{x})} \times ... \times t^{\mathcal{I}}_{m_j\,(\vec{x})}) \circ (\vec{\pi}))(P^{\mathcal{I}}_j) = \\
= \mathfrak{D}(t^{\mathcal{I}}_{1\,(\vec{x})} \times ... \times t^{\mathcal{I}}_{m_j\,(\vec{x})}) \circ \mathfrak{D}(\vec{\pi})(P^{\mathcal{I}}_j) = \\
= \mathfrak{D}(\vec{\pi}) \circ \mathfrak{D}(t^{\mathcal{I}}_{1\,(\vec{x})} \times ... \times t^{\mathcal{I}}_{m_j\,(\vec{x})})(P^{\mathcal{I}}_j) = \\
= \mathfrak{D}(\vec{\pi}) \circ (\mathcal{I}(P_j(t_1, ..., t_{m_j})_{(\vec{x})}))
\end{gathered}$$

- If $\varphi \equiv \psi \wedge \chi$, then by induction hypothesis and as $\mathfrak{D}(\vec{\pi})$ respects $\wedge$:

$$\begin{gathered}
\mathcal{I}(\psi \wedge \chi)_{(-,y,-)} = \\
= \mathcal{I}(\psi)_{(-,y,-)} \wedge \mathcal{I}(\chi)_{(-,y,-)} = \\
= \mathfrak{D}(\vec{\pi})(\mathcal{I}(\psi)_{(\vec{x})}) \wedge \mathfrak{D}(\vec{\pi})(\mathcal{I}(\chi)_{(\vec{x})}) = \\
= \mathfrak{D}(\vec{\pi})(\mathcal{I}(\psi)_{(\vec{x})} \wedge \mathcal{I}(\chi)_{(\vec{x})}) = \\
= \mathfrak{D}(\vec{\pi})(\mathcal{I}(\psi \wedge \chi)_{(\vec{x})})
\end{gathered}$$

- If $\varphi \equiv \psi \vee \chi$, then by induction hypothesis and as $\mathfrak{D}(\vec{\pi})$ respects $\vee$:

$$\begin{gathered}
\mathcal{I}(\psi \vee \chi)_{(-,y,-)} = \\
= \mathcal{I}(\psi)_{(-,y,-)} \vee \mathcal{I}(\chi)_{(-,y,-)} = \\
= \mathfrak{D}(\vec{\pi})(\mathcal{I}(\psi)_{(\vec{x})}) \vee \mathfrak{D}(\vec{\pi})(\mathcal{I}(\chi)_{(\vec{x})}) = \\
= \mathfrak{D}(\vec{\pi})(\mathcal{I}(\psi)_{(\vec{x})} \vee \mathcal{I}(\chi)_{(\vec{x})}) = \\
= \mathfrak{D}(\vec{\pi})(\mathcal{I}(\psi \vee \chi)_{(\vec{x})})
\end{gathered}$$

- If $\varphi \equiv \neg\psi$, then by induction hypothesis and as $\mathfrak{D}(\vec{\pi})$ respects $\neg$:

$$\begin{gathered}
\mathcal{I}(\neg\psi)_{(-,y,-)} = \\
= \neg\mathcal{I}(\psi)_{(-,y,-)} = \\
= \neg\mathfrak{D}(\vec{\pi})(\mathcal{I}(\psi)_{(\vec{x})}) = \\
= \mathfrak{D}(\vec{\pi})(\neg\mathcal{I}(\psi)_{(\vec{x})} = \\
= \mathfrak{D}(\vec{\pi})(\mathcal{I}(\neg\psi)_{(\vec{x})})
\end{gathered}$$

- If $\varphi \equiv \psi \to \chi$, then by induction hypothesis and as $\mathfrak{D}(\vec{\pi})$ respects $\to$:

$$\begin{gathered}\mathcal{I}(\psi \to \chi)_{(-,y,-)} = \\ = \mathcal{I}(\psi)_{(-,y,-)} \to \mathcal{I}(\chi)_{(-,y,-)} = \\ = \mathfrak{D}(\vec{\pi})(\mathcal{I}(\psi)_{(\vec{x})}) \to \mathfrak{D}(\vec{\pi})(\mathcal{I}(\chi)_{(\vec{x})}) = \\ = \mathfrak{D}(\vec{\pi})(\mathcal{I}(\psi)_{(\vec{x})} \to \mathcal{I}(\chi)_{(\vec{x})}) = \\ = \mathfrak{D}(\vec{\pi})(\mathcal{I}(\psi \to \chi)_{(\vec{x})})\end{gathered}$$

- If $\varphi \equiv \forall x_{n+1}.\psi(\vec{x}, x_{n+1})$, then by induction hypothesis and by the Beck-Chevalley conditions:

$$\begin{gathered}\mathcal{I}(\forall x_{n+1}.\psi(\vec{x}, x_{n+1}))_{(-,y,-,)} = \\ = \forall_{\pi_{n+1}} \mathcal{I}(\psi(\vec{x}, x_{n+1}))_{(-,y,-,x_{n+1})}) = \\ = \forall_{\pi_{n+1}}(\mathfrak{D}(\vec{\pi}, \pi_{n+1})(\mathcal{I}(\psi(\vec{x}, x_{n+1}))_{(\vec{x},x_{n+1})})) = \\ = \mathfrak{D}(\vec{\pi})(\forall_{\pi_{n+1}}(\mathcal{I}(\psi(\vec{x}, x_{n+1}))_{(\vec{x})})) = \\ = \mathfrak{D}(\vec{\pi})(\mathcal{I}(\forall x_{n+1}.\psi(\vec{x}, x_{n+1}))_{(\vec{x})}) =\end{gathered}$$

- If $\varphi \equiv \exists x_{n+1}.\psi(\vec{x}, x_{n+1})$, then by induction hypothesis and by the Beck-Chevalley conditions:

$$\begin{gathered}\mathcal{I}(\exists x_{n+1}.\psi(\vec{x}, x_{n+1}))_{(-,y,-,)} = \\ = \exists_{\pi_{n+1}} \mathcal{I}(\psi(\vec{x}, x_{n+1}))_{(-,y,-,x_{n+1})}) = \\ = \exists_{\pi_{n+1}}(\mathfrak{D}(\vec{\pi}, \pi_{n+1})(\mathcal{I}(\psi(\vec{x}, x_{n+1}))_{(\vec{x},x_{n+1})})) = \\ = \mathfrak{D}(\vec{\pi})(\exists_{\pi_{n+1}}(\mathcal{I}(\psi(\vec{x}, x_{n+1}))_{(\vec{x})})) = \\ = \mathfrak{D}(\vec{\pi})(\mathcal{I}(\exists x_{n+1}.\psi(\vec{x}, x_{n+1}))_{(\vec{x})}) =\end{gathered}$$

□

Lemma 8.2 (substitution). *Let $\mathcal{L}$ be a language for the intuitionistic predicative logic and let $\mathfrak{D}$ be a hyperdoctrine for the intuitionistic logic from a category $\mathcal{C}$. Let $t_{(x_1,\dots,x_n)}$ be a term and $\varphi_{(x_1,\dots,x_n)} \in Form_{\mathcal{L}}$ a formula, with free variables among $x_1, \dots x_n$. Then:*

(i) $\mathcal{I}(t_{(x_1,\dots,x_n)}[s/x_i]) = t^{\mathcal{I}} \circ (\pi_1 \times \dots \times s^{\mathcal{I}} \times \dots \times \pi_n)$

(ii) $\mathcal{I}(\varphi_{(x_1,\dots,x_n)}[s/x_i]) = \mathfrak{D}(\pi_1 \times \dots \times s^{\mathcal{I}} \times \dots \times \pi_n)(\mathcal{I}(\varphi)_{(x_1,\dots,x_n)})$

Proof. In the interests of brevity, we will write $(\vec{x}) \equiv (x_1, \dots, x_n)$ and $(-, s/x_i, -) \equiv (x_1/x_1, \dots, s/x_i, \dots, x_n/x_n)$, though we will refrain from writing the dependence of the formulas on free variables.

(i) Let's proceed by induction on the complexity of the term t.

- If $t \equiv x_i$, then:

$$\mathcal{I}(t_{(\vec{x})}[s/x_i]) = \mathcal{I}(x_i[s/x_i]) = s^{\mathcal{I}}$$

and

$$t^{\mathcal{I}} \circ (\pi_1 \times \dots \times s^{\mathcal{I}} \times \dots \times \pi_n) = x_i{}^{\mathcal{I}} \circ (\pi_1 \times \dots \times s^{\mathcal{I}} \times \dots \times \pi_n) = \pi_i \circ (\pi_1 \times \dots \times s^{\mathcal{I}} \times \dots \times \pi_n) = s^{\mathcal{I}}$$

If $t \equiv x_j$ with $j \neq i$, then:

$$\mathcal{I}(t_{(\vec{x})}[s/x_i]) = \mathcal{I}(x_j[s/x_i]) = \pi_j$$

and

$$t^{\mathcal{I}} \circ (\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n) = {x_j}^{\mathcal{I}} \circ (\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n) = \pi_j \circ (\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n) = \pi_j$$

- If $t \equiv f_j(t_1, ..., t_{m_j})$, then:

$$\begin{gathered}
\mathcal{I}(t_{(\vec{x})}[s/x_i]) = \\
= \mathcal{I}(f_j(t_1, ..., t_{m_j})[s/x_i]) = \\
= (f_j)^{\mathcal{I}} \circ (t_1[s/x_i]^{\mathcal{I}}, ..., t_{m_j}[s/x_i]^{\mathcal{I}}) = \\
= (f_j)^{\mathcal{I}} \circ (t_1^{\mathcal{I}}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n), ..., t_{m_j}^{\mathcal{I}}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)) = \\
= (f_j)^{\mathcal{I}} \circ ((t_1^{\mathcal{I}}, ..., t_{m_j}^{\mathcal{I}}) \circ (\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)) = \\
= (f_j)^{\mathcal{I}} \circ (t_1^{\mathcal{I}}, ..., t_{m_j}^{\mathcal{I}}) \circ (\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n) = \\
= (f_j(t_1, ..., t_{m_j}))^{\mathcal{I}} \circ (\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)
\end{gathered}$$

(ii) Let's proceed by induction of the complexity of the formula $\varphi \in Form_{\mathcal{L}}$.

- If $\varphi \equiv P_j(t_1, ..., t_{m_j})$, then by definition and by the previous point of the proof:

$$\begin{gathered}
\mathcal{I}(P_j(t_1, ..., t_{m_j})[s/x_i]) = \\
= \mathfrak{D}(t_1[s/x_i]^{\mathcal{I}} \times ... \times t_{m_j}[s/x_i]^{\mathcal{I}})(P_j^{\mathcal{I}}) = \\
= \mathfrak{D}(t_1^{\mathcal{I}} \circ (\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n) \times ... \times t_{m_j}^{\mathcal{I}} \circ (\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n))(P_j^{\mathcal{I}}) = \\
= \mathfrak{D}((t_1^{\mathcal{I}} \times ... \times t_{m_j}^{\mathcal{I}}) \circ (\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n))(P_j^{\mathcal{I}}) = \\
= \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n) \circ \mathfrak{D}(t_1^{\mathcal{I}} \times ... \times t_{m_j}^{\mathcal{I}})(P_j^{\mathcal{I}}) = \\
= \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(P_j(t_1, ..., t_{m_j})))
\end{gathered}$$

- If $\varphi \equiv \psi \wedge \chi$, then by definition and by induction hypothesis and as $\mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)$ respects $\wedge$:

$$\begin{gathered}
\mathcal{I}((\psi \wedge \chi)[s/x_i]) = \\
= \mathcal{I}(\psi[s/x_i]) \wedge \mathcal{I}(\chi[s/x_i]) = \\
= \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\psi)) \wedge \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\chi)) = \\
= \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\psi) \wedge \mathcal{I}(\chi)) = \\
= \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\psi \wedge \chi))
\end{gathered}$$

- If $\varphi \equiv \psi \vee \chi$, then by definition and by induction hypothesis and as $\mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)$ respects $\vee$:

$$\begin{gathered}
\mathcal{I}((\psi \vee \chi)[s/x_i]) = \\
= \mathcal{I}(\psi[s/x_i]) \vee \mathcal{I}(\chi[s/x_i]) = \\
= \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\psi)) \vee \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\chi)) = \\
= \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\psi) \vee \mathcal{I}(\chi)) = \\
= \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\psi \vee \chi))
\end{gathered}$$

- If $\varphi \equiv \neg\psi$, then by definition and by induction hypothesis and as $\mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)$ respects $\neg$:

$$\begin{gathered}\mathcal{I}((\neg\psi)[s/x_i]) = \\ = \neg\mathcal{I}(\psi[s/x_i]) = \\ = \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\neg\mathcal{I}(\psi)) = \\ = \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\neg\psi))\end{gathered}$$

- If $\varphi \equiv \psi \to \chi$, then by definition and by induction hypothesis and as $\mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)$ respects $\to$:

$$\begin{gathered}\mathcal{I}((\psi \to \chi)[s/x_i]) = \\ = \mathcal{I}(\psi[s/x_i]) \to \mathcal{I}(\chi[s/x_i]) = \\ = \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\psi)) \to \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\chi)) = \\ = \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\psi) \to \mathcal{I}(\chi)) = \\ = \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\psi \to \chi))\end{gathered}$$

- If $\varphi \equiv \forall x_{n+1}.\psi(\vec{x}, x_{n+1})$, then by definition and by induction hypothesis and by Beck-Chevalley conditions:

$$\begin{gathered}\mathcal{I}(\forall x_{n+1}.\psi(\vec{x}, x_{n+1})[s/x_i]) = \\ = \forall\pi_{n+1}(\mathcal{I}(\psi(\vec{x}, x_{n+1})[s/x_i])) = \\ = \forall\pi_{n+1}(\mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n \times \pi_{n+1})(\mathcal{I}(\psi(\vec{x}, x_{n+1})))) = \\ = \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n \times \pi_{n+1})(\forall\pi_{n+1}\mathcal{I}(\psi(\vec{x}, x_{n+1}))) = \\ = \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n \times \pi_{n+1})(\mathcal{I}(\forall x_{n+1}.\psi(\vec{x}, x_{n+1})))\end{gathered}$$

- If $\varphi \equiv \exists x_{n+1}.\psi(\vec{x}, x_{n+1})$, then by definition and by induction hypothesis and by Beck-Chevalley conditions:

$$\begin{gathered}\mathcal{I}(\exists x_{n+1}.\psi(\vec{x}, x_{n+1})[s/x_i]) = \\ = \exists\pi_{n+1}(\mathcal{I}(\psi(\vec{x}, x_{n+1})[s/x_i])) = \\ = \exists\pi_{n+1}(\mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n \times \pi_{n+1})(\mathcal{I}(\psi(\vec{x}, x_{n+1})))) = \\ = \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n \times \pi_{n+1})(\exists\pi_{n+1}\mathcal{I}(\psi(\vec{x}, x_{n+1}))) = \\ = \mathfrak{D}(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n \times \pi_{n+1})(\mathcal{I}(\exists x_{n+1}.\psi(\vec{x}, x_{n+1})))\end{gathered}$$

□

Lemma 8.3. *Let $\mathcal{L}$ be a language for the intuitionistic predicative logic and let $\mathfrak{D}$ be a hyperdoctrine for the intuitionistic logic from a category $\mathcal{C}$. Let's consider the assignation $c_i^{\mathcal{I}} = c_i$, $f_i^{\mathcal{I}} = f_i$ and $P_j(t_1, ..., t_{m_j})^{\mathcal{I}} = [P_j(t_1, ..., t_{m_j})]$. Then for every $\varphi_{(x_1,...,x_n)} \in Form_{\mathcal{L}}$ with free variables among $(x_1, ..., x_n)$:*

$$\mathcal{I}(\varphi)_{(x_1,...,x_n)} = [\varphi_{(x_1,...,x_n)}]$$

where $[\varphi_{(x_1,...,x_n)}]$ is the equivalent class of the proposition φ in the Lindenbaum algebra $\mathcal{A}(\mathcal{L})$ based on the language $\mathcal{L}$.

Proof. Let's proceed by induction on the complexity of the formula $\varphi \in Form_{\mathcal{L}}$.

- If $\varphi \equiv P_j(t_1, ..., t_{m_j})$, then $P_j(t_1, ..., t_{m_j})^{\mathcal{I}} = [P_j(t_1, ..., t_{m_j})]$ by hypothesis.

- If $\varphi \equiv \psi \wedge \chi$, then:

$$\begin{gathered}\mathcal{I}(\varphi)_{(x_1,\ldots,x_n)} = \\ = \mathcal{I}(\psi \wedge \chi)_{(x_1,\ldots,x_n)} = \\ = \mathcal{I}(\psi)_{(x_1,\ldots,x_n)} \wedge \mathcal{I}(\chi)_{(x_1,\ldots,x_n)} \overset{ind.hyp.}{=} \\ = [\psi_{(x_1,\ldots,x_n)}] \wedge [\chi_{(x_1,\ldots,x_n)}] = \\ = [(\psi \wedge \chi)_{(x_1,\ldots,x_n)}]\end{gathered}$$

- If $\varphi \equiv \psi \vee \chi$, then:

$$\begin{gathered}\mathcal{I}(\varphi)_{(x_1,\ldots,x_n)} = \\ = \mathcal{I}(\psi \vee \chi)_{(x_1,\ldots,x_n)} = \\ = \mathcal{I}(\psi)_{(x_1,\ldots,x_n)} \vee \mathcal{I}(\chi)_{(x_1,\ldots,x_n)} \overset{ind.hyp.}{=} \\ = [\psi_{(x_1,\ldots,x_n)}] \vee [\chi_{(x_1,\ldots,x_n)}] = \\ = [(\psi \vee \chi)_{(x_1,\ldots,x_n)}]\end{gathered}$$

- If $\varphi \equiv \neg\psi$, then:

$$\begin{gathered}\mathcal{I}(\varphi)_{(x_1,\ldots,x_n)} = \\ = \mathcal{I}(\neg\psi)_{(x_1,\ldots,x_n)} = \\ = \neg\mathcal{I}(\psi)_{(x_1,\ldots,x_n)} \overset{ind.hyp.}{=} \\ = \neg[\psi_{(x_1,\ldots,x_n)}] = \\ = [(\neg\psi)_{(x_1,\ldots,x_n)}]\end{gathered}$$

- If $\varphi \equiv \psi \to \chi$, then:

$$\begin{gathered}\mathcal{I}(\varphi)_{(x_1,\ldots,x_n)} = \\ = \mathcal{I}(\psi \to \chi)_{(x_1,\ldots,x_n)} = \\ = \mathcal{I}(\psi)_{(x_1,\ldots,x_n)} \to \mathcal{I}(\chi)_{(x_1,\ldots,x_n)} \overset{ind.hyp.}{=} \\ = [\psi_{(x_1,\ldots,x_n)}] \to [\chi_{(x_1,\ldots,x_n)}] = \\ = [(\psi \to \chi)_{(x_1,\ldots,x_n)}]\end{gathered}$$

- If $\varphi \equiv \forall x_{n+1}.\psi(x_{n+1})_{(x_1,\ldots,x_n)}$, then:

$$\begin{gathered}\mathcal{I}(\varphi)_{(x_1,\ldots,x_n)} = \\ = \mathcal{I}(\forall x_{n+1}.\psi(x_{n+1}))_{(x_1,\ldots,x_n)} = \\ = \forall\pi_{n+1}\mathcal{I}(\psi(x_{n+1}))_{(x_1,\ldots,x_n)} \overset{ind.hyp.}{=} \\ = \forall\pi_{n+1}[\psi((x_{n+1})_{(x_1,\ldots,x_n)}] = \\ = [(\forall x_{n+1}.\psi(x_{n+1}))_{(x_1,\ldots,x_n)}]\end{gathered}$$

- If $\varphi \equiv \exists x_{n+1}.\psi(x_{n+1})_{(x_1,\ldots,x_n)}$, then:

$$\begin{gathered}\mathcal{I}(\varphi)_{(x_1,\ldots,x_n)} = \\ = \mathcal{I}(\exists x_{n+1}.\psi(x_{n+1}))_{(x_1,\ldots,x_n)} = \\ = \exists\pi_{n+1}\mathcal{I}(\psi(x_{n+1}))_{(x_1,\ldots,x_n)} \overset{ind.hyp.}{=} \\ = \exists\pi_{n+1}[\psi((x_{n+1})_{(x_1,\ldots,x_n)}] = \\ = [(\exists x_{n+1}.\psi(x_{n+1}))_{(x_1,\ldots,x_n)}]\end{gathered}$$

$\square$

8.2 Theorem of Soundness and Completeness for predicate logic

Theorem 8.1 (Soundness). *Let $\mathcal{L}$ be a language for the (either intuitionistic or classical) predicative logic and let $Form_{\mathcal{L}}$ be the set of its formulas. Let $\Gamma_{(x_1,...,x_n)}, \varphi_{(x_1,...,x_n)} \in Form_{\mathcal{L}}$. If $\Gamma_{(x_1,...,x_n)} \vdash \varphi_{(x_1,...,x_n)}$, then for every hyperdoctrine $\mathfrak{D}$ and assignation $c_i^{\mathcal{I}}$, $f_i^{\mathcal{I}}$, $P_j^{\mathcal{I}}$: $\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$ in $\mathfrak{D}(C^n)$.*

Proof. We restrict to the case of the intuitionistic predicative logic, and we make use of the intuitionistic sequent calculus. Also, we omit the dependence of a formula on the free variables $x_1, ..., x_n$ in the notation. Let's suppose that π is a sequent calculus derivation tree for the sequent $\Gamma \vdash \varphi$ and let's proceed by induction on the complexity of the derivation tree π. Let $\mathfrak{D}$ be a hyperdoctrine.

Let's consider the base cases:

- The sequent $\Gamma \vdash \varphi$ is the axiom $id - ax$, thus $\Gamma \equiv \gamma_1, ..., \gamma_m, \varphi$. It follows that:

$$\mathcal{I}_{\mathfrak{D}}(\Gamma) = \mathcal{I}_{\mathfrak{D}}(\gamma_1 \wedge ... \wedge \gamma_m \wedge \varphi) = \mathcal{I}_{\mathfrak{D}}(\gamma_1) \wedge ... \wedge \mathcal{I}_{\mathfrak{D}}(\gamma_m) \wedge \mathcal{I}_{\mathfrak{D}}(\varphi) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$$

- The sequent $\Gamma \vdash \varphi$ is the axiom $\bot - ax$, thus $\Gamma \equiv \gamma_1, ..., \gamma_m, \bot$. It follows that:

$$\mathcal{I}_{\mathfrak{D}}(\Gamma) = \mathcal{I}_{\mathfrak{D}}(\gamma_1 \wedge ... \wedge \gamma_m \wedge \bot) = \mathcal{I}_{\mathfrak{D}}(\gamma_1) \wedge ... \wedge \mathcal{I}_{\mathfrak{D}}(\gamma_m) \wedge \mathcal{I}_{\mathfrak{D}}(\bot) = \mathcal{I}_{\mathfrak{D}}(\bot) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$$

- The sequent $\Gamma \vdash \varphi$ is the axiom $\top - ax$, thus $\varphi \equiv \top$. As a trivial consequence:

$$\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}}(\top) = \mathcal{I}_{\mathfrak{D}}(\varphi)$$

Let's consider the inductive steps:

- Let's suppose that the last step of π is the application of the $\wedge l$ rule, in case $\Gamma \equiv \Gamma', \alpha \wedge \beta$:

$$\dfrac{\begin{matrix} \vdots\, \pi' \\ \Gamma', \alpha, \beta \vdash \varphi \end{matrix}}{\Gamma', \alpha \wedge \beta \vdash \varphi}\ \wedge l$$

  By induction hypothesis we have that $\mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge \alpha \wedge \beta) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$, so by definition of interpretation:

$$\mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge \alpha \wedge \beta) = \mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\alpha \wedge \beta) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$$

- Let's suppose that the last step of π is the application of the $\wedge r$ rule, in case $\varphi \equiv \psi \wedge \chi$:

$$\dfrac{\begin{matrix} \vdots\, \pi' \\ \Gamma \vdash \psi \end{matrix} \qquad \begin{matrix} \vdots\, \pi'' \\ \Gamma \vdash \chi \end{matrix}}{\Gamma \vdash \psi \wedge \chi}\ \wedge r$$

  By induction hypothesis we have that $\mathcal{I}_{\mathfrak{D}}(\Gamma) \overset{(*)}{\leq} \mathcal{I}_{\mathfrak{D}}(\psi)$ and $\mathcal{I}_{\mathfrak{D}}(\Gamma) \overset{(**)}{\leq} \mathcal{I}_{\mathfrak{D}}(\chi)$, hence:

$$\mathcal{I}_{\mathfrak{D}}(\Gamma) \overset{(**)}{=} \mathcal{I}_{\mathfrak{D}}(\Gamma) \wedge \mathcal{I}_{\mathfrak{D}}(\chi) \overset{**}{\leq} \mathcal{I}_{\mathfrak{D}}(\psi) \wedge \mathcal{I}_{\mathfrak{D}}(\chi) = \mathcal{I}_{\mathfrak{D}}(\psi \wedge \chi) = \mathcal{I}_{\mathfrak{D}}(\varphi)$$

- Let's suppose that the last step of π is the application of the $\vee l$ rule, in case $\Gamma \equiv \Gamma', \alpha \vee \beta$:

$$\dfrac{\begin{matrix} \vdots\, \pi' \\ \Gamma', \alpha \vdash \varphi \end{matrix} \qquad \begin{matrix} \vdots\, \pi'' \\ \Gamma', \beta \vdash \varphi \end{matrix}}{\Gamma', \alpha \vee \beta \vdash \varphi}\ \vee l$$

By induction hypothesis we have that $\mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge \alpha) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$ and $\mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge \beta) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$. By definition of interpretation this means that $\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\alpha) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$ and $\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\beta) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$, which together can be written as

$$(\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\alpha)) \vee (\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\beta)) \leq \mathcal{I}_{\mathfrak{D}}(\varphi) \vee \mathcal{I}_{\mathfrak{D}}(\varphi) = \mathcal{I}_{\mathfrak{D}}(\varphi)$$

By distributivity and definition of interpretation the following follows:

$$\mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge (\alpha \vee \beta)) = \mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\alpha \vee \beta) = \mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge (\mathcal{I}_{\mathfrak{D}}(\alpha) \vee \mathcal{I}_{\mathfrak{D}}(\beta)) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$$

- Let's suppose that the last step of π is the application of the $\vee r_1$ rule (the rule $\vee r_2$ is similar), in case $\varphi \equiv \psi \vee \chi$:

$$\dfrac{\begin{matrix}\vdots\ \pi' \\ \Gamma \vdash \psi\end{matrix}}{\Gamma \vdash \psi \vee \chi}\ \vee r_1$$

By induction hypothesis we have that $\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}}(\psi)$, hence:

$$\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}}(\psi) \leq \mathcal{I}_{\mathfrak{D}}(\psi) \vee \mathcal{I}_{\mathfrak{D}}(\chi) = \mathcal{I}_{\mathfrak{D}}(\psi \vee \chi)$$

- Let's suppose that the last step of π is the application of the $\neg l$ rule, in case $\Gamma \equiv \Gamma', \neg\alpha$:

$$\dfrac{\begin{matrix}\vdots\ \pi' \\ \Gamma', \neg\alpha \vdash \alpha\end{matrix}}{\Gamma', \neg\alpha \vdash \varphi}\ \neg l$$

By induction hypothesis we have that $\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\neg\alpha) = \mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge \neg\alpha) \leq \mathcal{I}_{\mathfrak{D}}(\alpha)$, hence:

$$\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\neg\alpha) = \mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\neg\alpha) \wedge \mathcal{I}_{\mathfrak{D}}(\neg\alpha) \leq \mathcal{I}_{\mathfrak{D}}(\alpha) \wedge \mathcal{I}_{\mathfrak{D}}(\neg\alpha) = \mathcal{I}_{\mathfrak{D}}(\alpha \wedge \neg\alpha) \leq \mathcal{I}_{\mathfrak{D}}(\bot) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$$

- Let's suppose that the last step of π is the application of the $\neg r$ rule, in case $\varphi \equiv \neg\psi$:

$$\dfrac{\begin{matrix}\vdots\ \pi' \\ \Gamma, \psi \vdash\end{matrix}}{\Gamma \vdash \neg\psi}\ \neg r$$

By induction hypothesis we have that $\mathcal{I}_{\mathfrak{D}}(\Gamma) \wedge \mathcal{I}_{\mathfrak{D}}(\psi) = \mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge \psi) \leq \mathcal{I}_{\mathfrak{D}}(\bot)$, hence by definition of implication:

$$\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}}(\psi) \to \mathcal{I}_{\mathfrak{D}}(\bot) = \mathcal{I}_{\mathfrak{D}}(\psi \to \bot) \leq \mathcal{I}_{\mathfrak{D}}(\neg\psi)$$

- Let's suppose that the last step of π is the application of the $\to l$ rule, in case $\Gamma \equiv \Gamma', \alpha \to \beta$:

$$\dfrac{\begin{matrix}\vdots\ \pi' \\ \Gamma' \vdash \alpha\end{matrix} \qquad \begin{matrix}\vdots\ \pi'' \\ \Gamma', \beta \vdash \varphi\end{matrix}}{\Gamma', \alpha \to \beta \vdash \varphi}\ \to l$$

By induction hypothesis we have that $\mathcal{I}_{\mathfrak{D}}(\Gamma') \overset{(1)}{\leq} \mathcal{I}_{\mathfrak{D}}(\alpha)$ and $\mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge \beta) = \mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\beta) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$. By definition of implication, the last disequality can be expressed as $\mathcal{I}_{\mathfrak{D}}(\beta) \overset{(2)}{\leq} \mathcal{I}_{\mathfrak{D}}(\Gamma') \to \mathcal{I}_{\mathfrak{D}}(\varphi)$. Since in every Heyting algebra $x \wedge (x \to y) \leq y$ holds, we gather that:

$$\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge (\mathcal{I}_{\mathfrak{D}}(\alpha) \to \mathcal{I}_{\mathfrak{D}}(\beta)) \overset{(1)}{\leq} \mathcal{I}_{\mathfrak{D}}(\alpha) \wedge (\mathcal{I}_{\mathfrak{D}}(\alpha) \to \mathcal{I}_{\mathfrak{D}}(\beta)) \leq \mathcal{I}_{\mathfrak{D}}(\beta) \overset{(2)}{\leq} \mathcal{I}_{\mathfrak{D}}(\Gamma) \wedge \mathcal{I}_{\mathfrak{D}}(\varphi)$$

and again by definition of implication:

$$\mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge \alpha \to \beta) = \mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge (\mathcal{I}_{\mathfrak{D}}(\alpha) \to \mathcal{I}_{\mathfrak{D}}(\beta)) = \mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge (\mathcal{I}_{\mathfrak{D}}(\alpha) \to \mathcal{I}_{\mathfrak{D}}(\beta)) \wedge \mathcal{I}_{\mathfrak{D}}(\Gamma') \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$$

- Let's suppose that the last step of π is the application of the $\to r$ rule, in case $\varphi \equiv \psi \to \chi$:

$$\dfrac{\begin{array}{c}\vdots\ \pi' \\ \Gamma, \psi \vdash \chi\end{array}}{\Gamma \vdash \psi \to \chi} \to r$$

By induction hypothesis we have that $\mathcal{I}_{\mathfrak{D}}(\Gamma) \wedge \mathcal{I}_{\mathfrak{D}}(\psi) = \mathcal{I}_{\mathfrak{D}}(\Gamma \wedge \psi) \leq \mathcal{I}_{\mathfrak{D}}(\chi)$, hence by definition of implication:

$$\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}}(\psi) \to \mathcal{I}_{\mathfrak{D}}(\chi) = \mathcal{I}_{\mathfrak{D}}(\psi \to \chi)$$

- Let's suppose that the last step of π is the application of the $\forall l$ rule, in case $\Gamma \equiv \Gamma', \forall x_n.\alpha(x_1, ..., x_n)$:

$$\dfrac{\begin{array}{c}\vdots\ \pi' \\ \Gamma', \forall x_n.\alpha(x_1, ..., x_n), \alpha[t/x_n] \vdash \varphi\end{array}}{\Gamma', \forall x_n.\alpha(x_1, ..., x_n) \vdash \varphi} \forall l$$

By induction hypothesis we have that $\mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge \forall x_n.\alpha(x_1, ..., x_n) \wedge \alpha[t/x_n]) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$ and the following equivalences hold (due to the substitution Lemma (8.2) and the weakening Lemma (8.1)):

$$\begin{array}{c}
\mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge \forall x_n.\alpha(x_1, ..., x_n) \wedge \alpha[t/x_n]) \leq \mathcal{I}_{\mathfrak{D}}(\varphi) \\
\Updownarrow \\
\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\forall x_n.\alpha(x_1, ..., x_n)) \wedge \mathcal{I}_{\mathfrak{D}}(\alpha[t/x_n]) \leq \mathcal{I}_{\mathfrak{D}}(\varphi) \\
\Updownarrow \\
\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\forall x_n.\alpha(x_1, ..., x_n)) \wedge \mathfrak{D}(\pi_1 \times ... \times \pi_{n+1} \times t^{\mathcal{I}})(\mathcal{I}_{\mathfrak{D}}(\alpha(x_1, ..., x_n))) \leq \mathcal{I}_{\mathfrak{D}}(\varphi) \\
\Updownarrow \\
\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathfrak{D}(\pi_1 \times ... \times \pi_{n+1} \times t^{\mathcal{I}})(\mathcal{I}_{\mathfrak{D}}(\forall x_n.\alpha(x_1, ..., x_n)) \wedge \mathcal{I}_{\mathfrak{D}}(\alpha(x_1, ..., x_n))) \leq \mathcal{I}_{\mathfrak{D}}(\varphi) \\
\Updownarrow \\
\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathfrak{D}(\pi_1 \times ... \times \pi_{n+1} \times t^{\mathcal{I}})(\mathcal{I}_{\mathfrak{D}}(\forall x_n.\alpha(x_1, ..., x_n) \wedge \alpha(x_1, ..., x_n))) \leq \mathcal{I}_{\mathfrak{D}}(\varphi) \\
\Updownarrow \\
\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathfrak{D}(\pi_1 \times ... \times \pi_{n+1} \times t^{\mathcal{I}})(\mathcal{I}_{\mathfrak{D}}(\forall x_n.\alpha(x_1, ..., x_n))) \leq \mathcal{I}_{\mathfrak{D}}(\varphi) \\
\Updownarrow \\
\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\forall x_n.\alpha(x_1, ..., x_n)) \leq \mathcal{I}_{\mathfrak{D}}(\varphi) \\
\Updownarrow \\
\mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge \forall x_n.\alpha(x_1, ..., x_n)) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)
\end{array}$$

- Let's suppose that the last step of π is the application of the $\forall r$ rule, in case $\varphi \equiv \forall x_n.\psi(x_1, ..., x_n)$:

$$\dfrac{\begin{array}{c}\vdots\ \pi' \\ \Gamma \vdash \psi[w/x_n]\end{array}}{\Gamma \vdash \forall x_n.\psi(x_1, ..., x_n)} \forall r (w \notin FV(\Gamma, \forall x_n.\psi(x_1, ..., x_n)))$$

By induction hypothesis we have that $\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}}(\psi[w/x_n])$, hence by substitution Lemma (8.2): $\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathfrak{D}(\pi_1 \times ... \times \pi_{n-1} \times w^{\mathcal{I}})(\mathcal{I}_{\mathfrak{D}}(\psi))$. In addition, since $w \notin FV(\Gamma)$, by weakening Lemma (8.1) we have that $\mathfrak{D}(\pi_1 \times ... \times \pi_{n-1})(\mathcal{I}_{\mathfrak{D}}(\Gamma)_{(x_1,...,x_{n-1})}) = \mathcal{I}_{\mathfrak{D}}(\Gamma)_{(x_1,...,x_{n-1})}$. Finally, by the definition of hyperdoctrine there is an adjunction $\mathfrak{D}(\pi_n) \dashv \forall_{\pi_n}$ such that:

$$\begin{array}{c}
\mathcal{I}_{\mathfrak{D}}(\Gamma)_{(x_1,...,x_{n-1})} = \mathfrak{D}(\pi_1 \times ... \times \pi_{n-1})(\mathcal{I}_{\mathfrak{D}}(\Gamma)_{(x_1,...,x_{n-1})}) \leq \mathcal{I}_{\mathfrak{D}}(\psi[w/x_n]) \\
\Updownarrow \\
\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \forall_{\pi_n}\mathcal{I}_{\mathfrak{D}}(\psi(x_1, ..., x_n)) = \mathcal{I}_{\mathfrak{D}}(\forall x_n.\psi(x_1, ..., x_n))
\end{array}$$

- Let's suppose that the last step of π is the application of the $\exists l$ rule, in case $\Gamma \equiv \Gamma', \exists x_n.\alpha(x_1, ..., x_n)$:

$$\dfrac{\begin{matrix} \vdots \pi' \\ \Gamma', \alpha[w/x_n] \vdash \varphi \end{matrix}}{\Gamma', \exists x_n.\alpha(x_1, ..., x_n) \vdash \varphi} \ \exists l(w \notin FV(\Gamma', \exists x_n.\alpha(x_1, ..., x_n), \varphi))$$

By induction hypothesis we have that $\mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge \alpha[w/x_n]) = \mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\alpha[w/x_n]) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$, so by substitution Lemma (8.2): $\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathfrak{D}(\pi_1 \times ... \times \pi_{n-1} \times w^{\mathcal{I}})(\mathcal{I}_{\mathfrak{D}}(\alpha)) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$. Since $w \notin FV(\Gamma', \varphi)$, by applying weakening Lemma (8.1) we gather that $\mathcal{I}_{\mathfrak{D}}(\Gamma')_{(x_1,...,x_{n-1})} = \mathfrak{D}(\pi_1 \times ... \times \pi_{n-1})(\mathcal{I}_{\mathfrak{D}}(\Gamma')_{(x_1,...,x_{n-1})})$ and $\mathcal{I}_{\mathfrak{D}}(\varphi)_{(x_1,...,x_{n-1})} = D(\pi_1 \times ... \times \pi_{n-1})(\mathcal{I}_{\mathfrak{D}}(\varphi)_{(x_1,...,x_{n-1})})$. Using the properties of a Heyting algebra and the adjunction in the definition of hyperdoctrine, we deduce:

$$\mathfrak{D}(\pi_1 \times ... \times \pi_{n-1})(\mathcal{I}_{\mathfrak{D}}(\Gamma')) \wedge \mathcal{I}_{\mathfrak{D}}(\alpha[w/x_n]) \leq \mathfrak{D}(\pi_1 \times ... \times \pi_{n-1})(\mathcal{I}_{\mathfrak{D}}(\varphi))$$
$$\Updownarrow$$
$$\mathcal{I}_{\mathfrak{D}}(\alpha[w/x_n]) \leq \mathfrak{D}(\pi_1 \times ... \times \pi_{n-1})(\mathcal{I}_{\mathfrak{D}}(\Gamma')) \rightarrow \mathfrak{D}(\pi_1 \times ... \times \pi_{n-1})(\mathcal{I}_{\mathfrak{D}}(\varphi))$$
$$\Updownarrow$$
$$\mathcal{I}_{\mathfrak{D}}(\alpha[w/x_n]) \leq \mathfrak{D}(\pi_1 \times ... \times \pi_{n-1})(\mathcal{I}_{\mathfrak{D}}(\Gamma') \rightarrow \mathcal{I}_{\mathfrak{D}}(\varphi))$$
$$\Updownarrow$$
$$\exists_{\pi_n} \mathcal{I}_{\mathfrak{D}}(\alpha(x_1, ..., x_n)) \leq \mathcal{I}_{\mathfrak{D}}(\Gamma') \rightarrow \mathcal{I}_{\mathfrak{D}}(\varphi)$$
$$\Updownarrow$$
$$\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \exists_{\pi_n} \mathcal{I}_{\mathfrak{D}}(\alpha(x_1, ..., x_n)) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$$
$$\Updownarrow$$
$$\mathcal{I}_{\mathfrak{D}}(\Gamma') \wedge \mathcal{I}_{\mathfrak{D}}(\exists x_n.\alpha(x_1, ..., x_n)) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$$
$$\Updownarrow$$
$$\mathcal{I}_{\mathfrak{D}}(\Gamma' \wedge \exists x_n.\alpha(x_1, ..., x_n)) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$$

- Let's suppose that the last step of π is the application of the $\exists r$ rule, in case $\varphi \equiv \exists x_n.\psi(x_1, ..., x_n)$:

$$\dfrac{\begin{matrix} \vdots \pi' \\ \Gamma \vdash \psi[t/x_n] \end{matrix}}{\Gamma \vdash \exists x_n.\psi(x_1, ..., x_n)} \ \exists r$$

Trivially, $\exists_{\pi_n}(\mathcal{I}_{\mathfrak{D}}(\psi(x_1, ..., x_n)) \leq \exists_{\pi_n}(\mathcal{I}_{\mathfrak{D}}(\psi(x_1, ..., x_n))$, and due to the adjunction in the definition of hyperdoctrine, this is equivalent to:

$$\mathcal{I}_{\mathfrak{D}}(\psi(x_1, ..., x_n)) \leq \mathfrak{D}(\pi_1 \times ... \pi_{n-1})(\exists_{\pi_n} \mathcal{I}_{\mathfrak{D}}(\psi(x_1, ..., x_n)))$$

By applying $\mathfrak{D}(\pi_1 \times ... \times \pi_{n-1} \times t^{\mathcal{I}})$ to both sides, the following equivalences hold (owing to the substitution Lemma (8.2)):

$$\mathfrak{D}(\pi_1 \times ... \times \pi_{n-1} \times t^{\mathcal{I}})(\mathcal{I}_{\mathfrak{D}}(\psi(x_1, ..., x_n))) \leq \mathfrak{D}(\pi_1 \times ... \times \pi_{n-1} \times t^{\mathcal{I}})\mathfrak{D}(\pi_1 \times ... \pi_{n-1})(\exists_{\pi_n} \mathcal{I}_{\mathfrak{D}}(\psi(x_1, ..., x_n)))$$
$$\Updownarrow$$
$$\mathcal{I}_{\mathfrak{D}}(\psi[t/x_n]) \leq \mathfrak{D}((\pi_1 \times ... \times \pi_{n-1} \times t^{\mathcal{I}}) \circ (\pi_1 \times ... \pi_{n-1}))(\exists_{\pi_n} \mathcal{I}_{\mathfrak{D}}(\psi(x_1, ..., x_n)))$$
$$\Updownarrow$$
$$\mathcal{I}_{\mathfrak{D}}(\psi[t/x_n]) \leq \exists_{\pi_n} \mathcal{I}_{\mathfrak{D}}(\psi(x_1, ..., x_n))$$
$$\Updownarrow$$
$$\mathcal{I}_{\mathfrak{D}}(\psi[t/x_n]) \leq \mathcal{I}_{\mathfrak{D}}(\exists x_n.\psi(x_1, ..., x_n))$$

By induction hypothesis we have that $\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}}(\psi[t/x_n]) = \mathfrak{D}(\pi_1 \times ... \times \pi_n \times t^{\mathcal{I}})(\mathcal{I}(\psi(x_1, ..., x_n))$. Concluding: $\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}}(\psi[t/x_n]) \leq \mathcal{I}_{\mathfrak{D}}(\exists x_n.\psi(x_1, ..., x_n))$, hence: $\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}}(\exists x_n.\psi(x_1, ..., x_n))$.

□

Theorem 8.2 (Completeness). *Let $\mathcal{L}$ be a language for the (either intuitionistic or classical) predicative logic and let $Form_{\mathcal{L}}$ be the set of its formulas. Let $\Gamma_{(x_1,...,x_n)}, \varphi_{(x_1,...,x_n)} \in Form_{\mathcal{L}}$. If $\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$ in $\mathfrak{D}(C^n)$ for every hyperdoctrine $\mathfrak{D}$ and assignation $c_i^{\mathcal{I}}$, $f_i^{\mathcal{I}}$, $P_j^{\mathcal{I}}$, then: $\Gamma_{(x_1,...,x_n)} \vdash \varphi_{(x_1,...,x_n)}$.*

Proof. Similarly to the previous proof, we consider the case of the intuitionistic predicative logic. Let's suppose that for every hyperdoctrine $\mathfrak{D}$ and assignation $c_i^{\mathcal{I}}$, $f_i^{\mathcal{I}}$, $P_j^{\mathcal{I}}$ $\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$ holds. In particular, let's consider the identical assignation $c_i^{\mathcal{I}} = c_i$, $f_i^{\mathcal{I}} = f_i$ and $P_j(t_1, ..., t_{m_j})^{\mathcal{I}} = P_j(t_1, ..., t_{m_j})$. By Lemma (8.3), $\mathcal{I}$ is the identical interpretation, hence $\mathcal{I}_{\mathfrak{D}}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}}(\varphi)$ is equivalent to $[\Gamma] \leq [\varphi]$, which means that $\Gamma_{(x_1,...,x_n)} \vdash \varphi_{(x_1,...,x_n)}$. □

8.3 Adding the equality to the language: a new kind of interpretation

Let's consider a language $\mathcal{L}$ for the intuitionistic predicate logic with equality (we could as well naturally extend what is going to follow to the case of classical logic). We need to modify the interpretation of Definition (8.1), in order to be able to interpret statements with equality in a given Heyting algebra.

The most general idea is foreseeable. Let's consider a language $\mathcal{L}$ for the intuitionistic predicative logic with equality $IL_=$, where $(c_i)_{i\in I}$ are constants, $(f_i)_{i\in I}$ are terms and $(P_j)_{j\in J}$ are atomic predicates. Let $\mathcal{C}$ be a category with all finite products and let $\mathfrak{D}_=$ be a hyperdoctrine for $IL_=$. Given $C \in Ob(\mathcal{C})$, let be given an assignation:

- $(c_i)^{\mathcal{I}} : 1 \longrightarrow C$;
- $(f_i)^{\mathcal{I}} : C^{n_i} \longrightarrow C$ where n_i is the arity of f_i;
- $(P_j)^{\mathcal{I}} \in \mathfrak{D}_=(C^{n_j})$ where n_j is the arity of P_j, and in particular $=^{\mathcal{I}} \equiv \delta_C \in \mathfrak{D}_=(C^2)$ where δ_C is the fibred equality on the object C.

We define then the interpretation of terms and formulas based on the hyperdoctrine $\mathfrak{D}_=$ by induction on the construction of terms and formulas as in Definition (8.1), with the addition of the interpretation for formulas with the equality predicates:

$$\mathcal{I}_{\mathfrak{D}_=}((t = t')_{(x_1,...,x_n)}) := \mathfrak{D}_=(t^{\mathcal{I}} \times t'^{\mathcal{I}})(\delta_C)$$

The proof of the Theorem of Soundness and Completeness retraces exactly the same ideas of the previous sections of this chapter.

Having said that, we would like to tackle the proof of the Theorem of Soundness and Completeness for predicative logic with equality in a more specific way, enunciating it for some particular kinds of hyperdoctrines, namely the $\mathcal{H}$-valued sets (with $\mathcal{H}$ a given Heyting algebra). Indeed, those hyperdoctrines are really useful and handy when trying to construct counterexamples.

First of all, we recall that the equality is a binary equivalent relation, thus being reflexive, simmetric and transitive, which has to be evaluated in a given complete Heyting algebra (in the case of intuitionistic logic, which we decided to take under examination). For the sake of semplicity, let's suppose that the "objects"[2] over which we can express equality predicates are in a set A and let's also fix a complete Heyting algebra $\mathcal{H}$ in which we decide to evaluate the equality predicates. Then the most naive way to formalize equality is the following:

$$\begin{array}{rccl} =: & A \times A & \longrightarrow & \mathcal{H} \\ & (x,y) & \longmapsto & \begin{cases} \top & \text{if } x = y \\ \bot & \text{if } x \neq y \end{cases} \end{array} \tag{8.1}$$

[2] There is no categorical meaning here.

It is easy to convince ourselves that the foregoing construction captures the essence of equality, in some sense. Also, it can be quite easily expressed in a categorical-theoretic fashion, specifically by means of a hyperdoctrine:

$$\begin{array}{cccc} \mathfrak{D}_= : & \mathcal{C}^{op} & \longrightarrow & \mathbf{Ha} \\ & C & \longmapsto & \mathcal{H}^C \\ & (f : C \longrightarrow D) & \longmapsto & (- \circ f : \mathcal{H}^D \longrightarrow \mathcal{H}^C) \end{array}$$

where $\mathcal{H}^C$ indicates the collection of all maps from the object $C \in Ob(\mathcal{C})$ to the Heyting algebra $\mathcal{H}$, and

$$\begin{array}{cccc} - \circ f : & \mathcal{H}^D & \longrightarrow & \mathcal{H}^C \\ & (D \xrightarrow{\varphi} \mathcal{H}) & \longmapsto & (C \xrightarrow{f} D \xrightarrow{\varphi} \mathcal{H}) \end{array}$$

It is possible now to extend Lemma (8.1), Lemma (8.2), Lemma (8.3), Theorem (8.1) and Theorem (8.2) to the case of predicative logic with equality, provided that the interpretation of Definition (8.1) is properly adjusted.

Nevertheless, we do not follow this strategy, opting for a more subtle one, since construction (8.1) has a little drawback: it is somehow "too classical". Indeed, according to it, the truth value of an equality predicate could only be either $\top$ or $\bot$: this is fine in the case of classical logic, but does not go hand-in-hand with the more complex intuitionistic expressive power. Finding a way to espress the typical intuitionistic expressive power is possible, but it requires a more ingenious procedure, namely the use of $\mathcal{H}$-valued equivalence relations and $\mathcal{H}$-valued predicates.

Let's fix a complete Heyting algebra $\mathcal{H}$.

Definition 8.2. *Let $\mathcal{H}$ be a Heyting algebra and A be a non-empty set. A function*

$$\rho : A \times A \longrightarrow \mathcal{H}$$

*is said to be a $\mathcal{H}$-**valued equivalence relation** if the following conditions are satified:*

- *(reflexivity) for every $a, a' \in A$*

$$\rho(a, a') = 1 \Leftrightarrow a = a' \in A$$

- *(symmetricity) for every $A, a' \in A$*

$$\rho(a, a') \leq {}^3\rho(a', a)$$

- *(transitivity) for every $a, a', a'' \in A$*

$$\rho(a, a') \wedge \rho(a', a'') \leq {}^4\rho(a, a'')$$

Definition 8.3. *Let $\mathcal{H}$ be a Heyting algebra, A be a non-empty set and $\rho : A \times A \longrightarrow \mathcal{H}$ be a $\mathcal{H}$-valued equivalence relation. A function*

$$P : A \longrightarrow \mathcal{H}$$

*is said to be a $\mathcal{H}$-**valued predicate with respect to** ρ **on** A if the following holds:*

- *(substitutivity) for every $a, a' \in A$*

$$\rho(a, a') \wedge P(a) \leq P(a')$$

[3]The condition $\rho(a, a') = \rho(a', a)$ is trivially equivalent.

[4]One may wonder why not to ask $\rho(a, a') \wedge \rho(a', a'') = \rho(a, a'')$. We do not impose this condition otherwise every element is in the relation ρ with any other element since: $\rho(a, a') \wedge \rho(a', a) = \rho(a, a)$

Definition 8.4. *Let $\mathcal{H}$ be a Heyting algebra, A, A' be non-empty sets endowed with $\mathcal{H}$-valued equivalence relations $\rho : A \times A \longrightarrow \mathcal{H}$ and $\rho' : A' \times A' \longrightarrow \mathcal{H}$. A function*

$$f : A \longrightarrow A'$$

is said to be a ***$\mathcal{H}$-valued function with respect to ρ, ρ' on*** *A, A' if the following holds:*

- *(substitutivity) for every $a, a' \in A$*

$$\rho(a, a') \leq \rho'(f(a), f(a'))$$

It can be shown that, given a $\mathcal{H}$-valued equivalence relation on a set, the natural extension of the definition to the cartesian product gives rise to a $\mathcal{H}$-valued equivalence relation on the cartesian product, where projections and pairing are $\mathcal{H}$-valued functions. Also, the set of all n-ary $\mathcal{H}$-valued predicates, with the punctual order, is a Heyting algebra; it is complete if $\mathcal{H}$ is; it is a Boolean algebra if $\mathcal{H}$ is.

Usually, the foregoing notions are used in the algebraic-theoretic proof of the Theorem of Soundness of predicate logic with equality, interpreting predicates as $\mathcal{H}$-valued predicates, functions as $\mathcal{H}$-valued functions and equality as a $\mathcal{H}$-valued equivalence relation ρ, after a complete Heyting (Boolean, in the classical case) $\mathcal{H}$ is fixed.

The idea now is to stick to the same plan, though proceeding in a more general categorical-theoretic fashion. Let's fix a complete Heyting algebra $\mathcal{H}$. We define the following new category:

Definition 8.5. *The category $\mathcal{H} - \mathbf{Set}$ is defined as follows:*

objects *an object is a couple (C, ρ) where $C \in Ob(\mathbf{Set})$ and $\rho : C \times C \longrightarrow \mathcal{H}$ is such that for every $c, c', c'' \in C$:*

- *(reflexivity) $\rho(c, c) = \top$;*
- *(simmetricity) $\rho(c, c') \leq \rho(c', c)$;*
- *(transitivity) $\rho(c, c') \wedge \rho(c', c'') \leq \rho(c, c'')$.*

maps *a map $[f] : (C, \rho) \longrightarrow (D, \sigma)$ is an equivalent class of functions $f : C \longrightarrow D$ such that for every $c, c' \in C$:*

- *(substitutivity) $\rho(c, c') \leq \sigma(f(c), f(c'))$*

At this point, we can define a new hyperdoctrine that will be used to define a new kind of interpretation as follows:

$$\begin{array}{cccc} \mathfrak{D}_= : & (\mathcal{H} - \mathbf{Set})^{op} & \longrightarrow & \mathbf{Ha} \\ & (C, \rho) & \longmapsto & \{\varphi \in \mathcal{H}^C \mid \forall c, c' \in C.(\varphi(c) \wedge \rho(c, c') \leq \varphi(c'))\} \\ & ([f] : (C, \rho) \longrightarrow (D, \sigma)) & \longmapsto & (- \circ [f] : \mathfrak{D}_=(D, \sigma) \longrightarrow \mathfrak{D}_=(C, \rho)) \end{array}$$

Before introducing the Definition (8.6) of interpretation for predicative logic with equality in the particular case we are studying, we need to verify that the functor $\mathfrak{D}_=$ is indeed a hyperdoctrine, thus we need to show that the category $(\mathcal{H} - \mathbf{Set})^{op}$ has all finite products (it suffices to show it has all binary products), that for every projection $\pi_i : (C_1, \rho_1) \times (C_2, \rho_2) \longrightarrow (C_i, \rho_i)$ in $(\mathcal{H} - \mathbf{Set})^{op}$ there are a left adjoint $\exists_{\pi_i}$ and a right adjoint $\forall_{\pi_i}$ to $\mathfrak{D}_=(\pi_i)$ respecting the Beck-Chevalley and Frobenius conditions as requested by Definition (7.4), that for every object $(C, \rho) \in Ob((\mathcal{H} - \mathbf{Set})^{op})$ there exists the fibred equality $\delta_{(C,\rho)}$ on the object (C, ρ) as requested by Definition (7.6). For the sake of semplicity, we will write objects (C, ρ) of $(\mathcal{H} - \mathbf{Set})^{op}$ without specifying the $\mathcal{H}$-valued equivalence relation ρ associated to the set C, unless it is necessary.

First of all, let's consider two objects (C_1, ρ_1) and (C_2, ρ_2) in $(\mathcal{H} - \mathbf{Set})^{op}$. Their binary product is the object $(C_1 \times C_2, \rho_1 \times \rho_2)$, where $C_1 \times C_2$ is the cartesian product of the sets C_1 and C_2 and

$$\begin{array}{cccc} \rho_1 \times \rho_2 : & (C_1 \times C_2) \times (C_1 \times C_2) & \longrightarrow & \mathcal{H} \\ & ((c_1, c_2), (c_1', c_2')) & \longmapsto & \rho_1(c_1, c_1') \wedge \rho_2(c_2, c_2') \end{array}$$

Trivially, $\rho_1 \times \rho_2$ respects reflexivity, simmetricity, transitivity and substitutivity, as Definition (8.5) requires, since ρ_1 and ρ_2 do:

- $\rho_1(c_1,c_1) \wedge \rho_2(c_2,c_2) = \top \wedge \top = \top$ for every $c_1 \in C_1$ and $c_2 \in C_2$;
- $\rho_1(c_1,c_1') \wedge \rho_2(c_2,c_2') \leq \rho_1(c_1',c_1) \wedge \rho_2(c_2,c_2')$ for every $c_1,c_1' \in C_1$ and $c_2,c_2' \in C_2$;
- $(\rho_1(c_1,c_1') \wedge \rho_2(c_2,c_2')) \wedge (\rho_1(c_1',c_1'') \wedge \rho_2(c_2',c_2'')) \leq (\rho_1(c_1,c_1') \wedge \rho_1(c_1',c_1'')) \wedge (\rho_2(c_2,c_2') \wedge \rho_2(c_2',c_2'')) \leq \rho_1(c_1,c_1'') \wedge \rho_2(c_2,c_2''))$ for every $c_1,c_1',c_1'' \in C_1$ and $c_2,c_2'.c_2'' \in C_2$;
- for any map $[f] : (C_1 \times C_2, \rho_1 \times \rho_2) \longrightarrow (D,\sigma)$ in $\mathcal{H}-\mathbf{Set}$ and for every $c_1,c_1' \in C_1$ and $c_2,c_2' \in C_2$: $\rho_1(c_1,c_1') \wedge \rho_2(c_2,c_2') \leq \sigma(\bigwedge_{c_2\in C_2} f(c_1,c_2), \bigwedge_{c_2\in C_2} f(c_1',c_2)) \wedge \sigma(\bigwedge_{c_1\in C_1} f(c_1,c_2), \bigwedge_{c_1\in C_1} f(c_1,c_2')) \wedge \leq \sigma(f(c_1,c_2), f(c_1',c_2'))$

The projections $\pi_i : C_1 \times C_2 \longrightarrow C_i$ are the canonical ones, and they really are maps in $\mathcal{H}-\mathbf{Set}$ since the condition of substitutivity in Definition (8.5) is respected, as we can see by replacing f with π_i in the last entry of the foregoing bulleted list. Also, for every object X in $\mathcal{H}-\mathbf{Set}$ and diagram of the type

$$\begin{tikzcd} & X \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] & \\ & C_1 \times C_2 \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ C_1 & & C_2 \end{tikzcd}$$

there is a unique map $\overline{f} : X \longrightarrow C_1 \times C_2$, namely $\overline{f} \equiv (f_1,f_2)$, such that the preceding diagram commutes.

Let $\pi_1 : C_1 \times C_2 \longrightarrow C_1$ be a projection (the proof for $\pi_2 : C_1 \times C_2 \longrightarrow C_i$ is similar). We recall that the image of the map π_1 through the functor $\mathfrak{D}_=$ gives rise to the map

$$\begin{array}{llcl} \mathfrak{D}_= : & \mathfrak{D}_=(C_1) = \mathcal{H}^{C_1} & \longrightarrow & \mathfrak{D}_=(C_1 \times C_2) = \mathcal{H}^{C_1\times C_2} \\ & (\varphi : C_1 \longrightarrow \mathcal{H}) & (\varphi \circ \pi_1 : C1 \times C_2 \longrightarrow \mathcal{H}) & \end{array}$$

We define its left and right adjoints as follows:

$$\begin{array}{lccc} \exists_{\pi_1} : & \mathfrak{D}_=(C_1 \times C_2) = \mathcal{H}^{C_1\times C_2} & \longrightarrow & \mathfrak{D}_=(C_1) = \mathcal{H}^{C_1} \\ & (\psi : C_1 \times C_2 \longrightarrow \mathcal{H}) & \longrightarrow & (\exists x_2.\psi : C_1 \longrightarrow \mathcal{H}) \end{array}$$

where in particular $(\exists x_2.\psi)(c_1) \coloneqq \bigvee_{x\in C_2} \psi(c_1,x)$ for every $c_1 \in C_1$, and

$$\begin{array}{lccc} \forall_{\pi_1} : & \mathfrak{D}_=(C_1 \times C_2) & \longrightarrow & \mathfrak{D}_=(C_1) \\ & \mathcal{H}^{C_1\times C_2} & \longrightarrow & \mathcal{H}^{C_1} \\ & (\psi : C_1 \times C_2 \longrightarrow \mathcal{H}) & \longmapsto & (\forall x_2.\psi : C_1 \longrightarrow \mathcal{H}) \end{array}$$

where in particular $(\exists x_2.\psi)(c_1) \coloneqq \bigwedge_{x\in C_2} \psi(c_1,x)$ for every $c_1 \in C_1$. They indeed are left and right adjoints to $\mathfrak{D}_=(\pi_1)$, since for every $\varphi \in \mathcal{H}^{C_1}, \psi \in \mathcal{H}^{C_1\times C_2}$:

$$(\exists x_2.\psi)(c_1) \leq \varphi(c_1) \Leftrightarrow \psi(c_1,c_2) \leq \varphi(\pi_1(c_1,c_2)) \text{ for every } c_1 \in C_1, c_2 \in C_2$$

that is

$$\bigvee_{x\in C_2} \psi(c_1,x) \leq \varphi(c_1) \Leftrightarrow \psi(c_1,c_2) \leq \varphi(c_1) \text{ for every } c_1 \in C_1, c_2 \in C_2$$

and

$$\varphi(\pi_1(c_1,c_2)) \leq \psi(c_1,c_2) \Leftrightarrow \varphi(c_1) \leq (\forall x_2.\psi)(c_1) \text{ for every } c_1 \in C_1, c_2 \in C_2$$

that is

$$\varphi(c_1) \leq \psi(c_1,c_2) \Leftrightarrow \varphi(c_1) \leq \bigwedge_{x\in C_2} \psi(c_1,x) \text{ for every } c_1 \in C_1, c_2 \in C_2$$

The left adjoint $\exists_{\pi_1}$ respects the Beck-Chevalley condition: given two projections $\pi : X \times C \longrightarrow C$, $\pi' : X' \times C' \longrightarrow C'$ (without loss of generality we can reduce to projections of this form) and a pullback diagram

$$\begin{array}{ccc} X' \times C' & \xrightarrow{\pi'} & C' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X \times C & \xrightarrow[\pi]{} & C \end{array}$$

the map $\exists_{\pi'}\mathfrak{D}_=(f')(\beta) \leq \mathfrak{D}_=(f)\exists_\pi(\beta)$ is an isomorphism for every $\beta \in \mathfrak{D}_=(X \times C) = \mathcal{H}^{X\times C}$, because

$$\beta \in \mathcal{H}^{X\times C} \underset{\mathfrak{D}_=(f')}{\longmapsto} \beta \circ f' \in \mathcal{H}^{X'\times C'} \underset{\exists_{\pi'}}{\longmapsto} \bigvee_{x'\in X'} \beta(f'(x',-)) = \bigvee_{x\in X, x'\in X'} \beta(x, (\pi \circ f')(x',-)) \in \mathcal{H}^{C'}$$

$$\beta \in \mathcal{H}^{X\times C} \underset{\exists_\pi}{\longmapsto} \bigvee_{x\in X} \beta(x,c) \in \mathcal{H}^{C} \underset{\mathfrak{D}_=(f)}{\longmapsto} \bigvee_{x\in X} \beta(x, f(-)) = \bigvee_{x\in X, x'\in X'} \beta(x, (f \circ \pi')(x',-)) \in \mathcal{H}^{C'}$$

and recalling that $f \circ \pi' = \pi \circ f'$ owing to the Definition (3.8) of pullback, we conclude that $\exists_{\pi'}\mathfrak{D}_=(f')(\beta) = \mathfrak{D}_=(f)\exists_\pi(\beta)$.

It respects also the Frobenius condition: given a projection $\pi : X \times C \longrightarrow C$, then the map $\exists_\pi(\mathfrak{D}_=(\pi)(\alpha) \wedge \beta) \leq \alpha \wedge \exists_\pi(\beta)$ is an isomorphism for every $\alpha \in \mathcal{H}^C$ and $\beta \in \mathcal{H}^{X\times C}$, because the following holds for every $x \in X, c \in C$:

$$\exists_\pi(\mathfrak{D}_=(\pi)(\alpha) \wedge \beta)(x,c) = (\alpha \times \pi)(x,c) \wedge \beta(x,c) = \bigvee_{x\in X} (\alpha \circ \pi)(x,c) \wedge \beta(x,c) =$$
$$\alpha(c) \wedge \bigvee_{x\in X} \beta(x,c) = \alpha(c) \wedge (\exists x.\beta)(c) = \alpha(c) \wedge \exists_\pi(\beta)(c)$$

The right adjoint $\forall_{\pi_1}$ respects the Beck-Chevalley condition: given two projections $\pi : X\times C \longrightarrow C$, $\pi' : X' \times C' \longrightarrow C'$ and a pullback diagram

$$\begin{array}{ccc} X' \times C' & \xrightarrow{\pi'} & C' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X \times C & \xrightarrow[\pi]{} & C \end{array}$$

the map $\mathfrak{D}_=(f)\forall_\pi(\beta) \leq \forall_{\pi'}\mathfrak{D}_=(f')(\beta)$ is an isomorphism for every $\beta \in \mathfrak{D}_=(X \times C) = \mathcal{H}^{X\times C}$, because

$$\beta \in \mathcal{H}^{X\times C} \underset{\forall_\pi}{\longmapsto} \bigwedge_{x\in X} \beta(x,c) \in \mathcal{H}^{C} \underset{\mathfrak{D}_=(f)}{\longmapsto} \bigwedge_{x\in X} \beta(x, f(-)) = \bigwedge_{x\in X, x'\in X'} \beta(x, (f \circ \pi')(x',-)) \in \mathcal{H}^{C'}$$

$$\beta \in \mathcal{H}^{X\times C} \underset{\mathfrak{D}_=(f')}{\longmapsto} \beta \circ f' \in \mathcal{H}^{X'\times C'} \underset{\forall_{\pi'}}{\longmapsto} \bigwedge_{x'\in X'} \beta(f'(x',-)) = \bigwedge_{x\in X, x'\in X'} \beta(x, (\pi \circ f')(x',-)) \in \mathcal{H}^{C'}$$

and recalling that $f \circ \pi' = \pi \circ f'$ owing to the Definition (3.8) of pullback, we conclude that $\mathfrak{D}_=(f)\forall_\pi(\beta) = \forall_{\pi'}\mathfrak{D}_=(f')(\beta)$.

Finally, let (C,ρ) be an object in $(\mathcal{H} - \mathbf{Set})^{op}$. We define the fibred equality on (C,ρ) by:

$$\delta_{(C,\rho)} \coloneqq \rho \in Ob(\mathfrak{D}_=((C,\rho) \times (C,\rho)))$$

where ρ is indeed an object of $\mathfrak{D}_=((C,\rho) \times (C,\rho))$ because $\rho \in \mathcal{H}^{C\times C}$ and $\rho(c,c') \wedge \rho(c,d) \wedge \rho(c',d') \leq \rho(d,d')$ thanks to simmetricity and transitivity of ρ in Definition (8.5). Let's consider any $X \in Ob((\mathcal{H} - \mathbf{Set})^{op})$ and the map $(\pi_1, \pi_2\pi_2) : X \times C \longrightarrow X \times C \times C$ in $(\mathcal{H} - \mathbf{Set})^{op}$. The following functors give rise to an adjunction with left adjoint:

$$\begin{array}{rccc} \exists_{(\pi_1,\pi_2,\pi_2)} : & \mathfrak{D}_=(X \times C) = \mathcal{H}^{X\times C} & \longrightarrow & \mathfrak{D}_=(X \times C \times C) = \mathcal{H}^{X\times C\times C} \\ & \varphi & \longmapsto & \mathfrak{D}_=(1_X \times \pi_2)(\varphi) \wedge \mathfrak{D}_=(\pi_2,\pi_2)(\delta_C) \end{array}$$

where $\mathfrak{D}_=(1_X \times \pi_2)(\varphi) \wedge \mathfrak{D}_=(\pi_2,\pi_2)(\delta_C) = \varphi \circ (1_X \times \pi_2) \wedge \delta_C \circ (\pi_2,\pi_2) = \varphi(x,c) \wedge \rho(c,c')$ for all $(x,c,c') \in X \times C \times C$, and with right adjoint:

$$\begin{array}{rccc} \mathfrak{D}_=(\pi_1,\pi_2,\pi_2) : & \mathfrak{D}_=(X \times C \times C) = \mathcal{H}^{X\times C\times C} & \longrightarrow & \mathfrak{D}_=(X \times C) = \mathcal{H}^{X\times C} \\ & \psi & \longmapsto & \psi \circ (\pi_1,\pi_2,\pi_2) \end{array}$$

This actually is an adjunction because for every $\varphi \in \mathcal{H}^{X\times C}, \psi \in \mathcal{H}^{X\times C\times C}$ the following holds for every $x \in X$ and $c, c' \in C$:

$$\varphi(x,c) \wedge \rho(c,c') \leq \psi(x,c,c') \Leftrightarrow \varphi(x,c) \leq (\psi \circ (\pi_1,\pi_2,\pi_2))(x,c) = \psi(x,c,c)$$

as it can be easily shown:

($\Rightarrow$) If $\varphi(x,c) \leq \psi(x,c,c)$, then we gather:

$$\varphi(x,c) \wedge \rho(c,c') \leq \psi(x,c,c) \wedge \rho(c,c')$$

and due the way $\mathfrak{D}_=$ is defined:

$$\varphi(x,c) \wedge \rho(c,c') \leq \psi(x,c,c) \wedge \rho(c,c') \leq \psi(x,c,c')$$

($\Leftarrow$) Let's suppose that $\varphi(x,c) \wedge \rho(c,c') \leq \psi(x,c,c')$ holds. It follows that:

$$\varphi(x,c) \wedge \rho(c,c') \wedge \rho(c',c) \leq \psi(x,c,c') \wedge \rho(c',c)$$

and by the definition of implication in a Heyting algebra:

$$\varphi(x,c) \leq \rho(c,c') \wedge \rho(c',c) \rightarrow \psi(x,c,c') \wedge \rho(c',c)$$

However, by transitivity and reflexivity of ρ (Definition (8.5)) $\rho(c,c') \wedge \rho(c',c) \leq \rho(c,c) = \top$, so:

$$\varphi(x,c) \leq \rho(c,c) \rightarrow \psi(x,c,c') \wedge \rho(c',c)$$

and again due to the fact that we are working in a Heyting algebra:

$$\varphi(x,c) \wedge \rho(c,c) \leq \psi(x,c,c') \wedge \rho(c',c)$$

hence, using again the definition of $\mathfrak{D}_=$, we can conclude:

$$\varphi(x,c) = \varphi(x,c) \wedge \top = \varphi(x,c) \wedge \rho(c,c) \leq \psi(x,c,c') \wedge \rho(c',c) \leq \psi(x,c,c)$$

In the end, we have proved that $\mathfrak{D}_=$ is in fact a hyperdoctrine for the intuitionistic predicative logic with equality $IL_=$, and we can now stick to the plan we announced some lines above.

Definition 8.6. *Let $\mathcal{L}$ be a language for the (intuitionistic or classical) predicative logic with equality $=$, let $(c_i)_{i\in I}$ be symbols for constants, $(f_i)_{i\in I}$ be symbols for terms, $(P_j)_{j\in J}$ be symbols for predicates. Let $\mathcal{H}$ be a complete Heyting algebra and let $\mathfrak{D}_=$ be the hyperdoctrine as we defined it above. Given $(C,\rho) \in Ob(\mathcal{H} - \mathbf{Set})$, let be given an assignation:*

- $(c_i)^{\mathcal{I}} : (1,\delta) \longrightarrow (C,\rho)$, *where*
$$\begin{array}{rccc} \delta: & 1\times 1 & \longrightarrow & \mathcal{H} \\ & (\cdot,\cdot) & \longmapsto & \top \end{array}$$
- $(f_i)^{\mathcal{I}} : (C,\rho)^{n_i} \longrightarrow (C,\rho)$, *where n_i is the arity of f_i and $(C,\rho)^{n_i} := (C^{n_i},\rho^{n_i})$ with*
$$\begin{array}{rccc} \rho^{n_i}: & C^{n_i}\times C^{n_i} & \longrightarrow & \mathcal{H} \\ & (c_1,...c_{n_i},\widetilde{c_1},...,\widetilde{c_{n_i}}) & \longmapsto & \rho(c_1,\widetilde{c_1}) \wedge ... \wedge \rho(c_{n_i},\widetilde{c_{n_i}}) \end{array}$$
- $(P_j)^{\mathcal{I}} \in \mathfrak{D}_=((C,\rho)^{n_j}) = \{\varphi \in \mathcal{H}^{C^{n_j}} | \varphi(c_1,...,c_{n_j}) \wedge \rho(c_1,c_1') \wedge ... \wedge \rho(c_{n_j},c_{n_j}') \leq \varphi(c_1',...,c_{n_j}')\}$, *where n_j is the arity of P_j.*

We define the ***interpretation of terms and formulas based on the hyperdoctrine*** $\mathfrak{D}_=$ *by induction on the construction of terms and formulas.*

The ***term interpretation*** *of a term $t_{(x_1,...,x_n)}$, with free variables among $x_1,...,x_n$, is a map*

$$t^{\mathcal{I}} : (C,\rho)^n \longrightarrow (C,\rho)$$

in $\mathcal{C}$, defined by induction as follows:

- $\mathcal{I}(x_{i(x_1,...,x_n)}) \coloneqq \pi_i : (C,\rho)^n \longrightarrow (C,\rho)$ *with* π_i *projection*
- $\mathcal{I}(f_i(t_1,...,t_m)_{(x_1,...,x_n)}) \coloneqq (f_i)^{\mathcal{I}}(\mathcal{I}(t_1),...,\mathcal{I}(t_m))$

The ***formula interpretation*** *of a formula* $\varphi_{(x_1,...,x_n)}$*, with free variables among* $x_1,...,x_n$*, is the assignation*

$$\mathcal{I}(\varphi_{(x_1,...,x_n)}) \in \mathfrak{D}_{=}((C,\rho)^n)$$

defined by induction on the complexity of the formula $\varphi(x_1,...,x_n)$ *as follows:*

- $\mathcal{I}(P_j(t_1,...,t_{m_j})_{(x_1,...,x_n)}) \coloneqq \mathfrak{D}_{=}(t_1^{\mathcal{I}} \times ... \times t_{m_j}^{\mathcal{I}})(P_j^{\mathcal{I}})$
- $\mathcal{I}((t = t')_{(x_1,...,x_n)}) \coloneqq \mathfrak{D}_{=}(t^{\mathcal{I}} \times t'^{\mathcal{I}})(\rho)$ *with* t, t' *terms and* $\rho \in \mathfrak{D}_{=}((C,\rho)^2)$[5]
- $\mathcal{I}(\neg\psi)_{(x_1,...,x_n)} \coloneqq \neg\mathcal{I}(\psi)_{(x_1,...,x_n)} \in \mathfrak{D}_{=}((C,\rho)^n)$
- $\mathcal{I}(\psi \wedge \chi)_{(x_1,...,x_n)} \coloneqq \mathcal{I}(\psi_{(x_1,...,x_n)}) \wedge \mathcal{I}(\chi_{(x_1,...,x_n)}) \in \mathfrak{D}_{=}((C,\rho)^n)$
- $\mathcal{I}(\psi \vee \chi)_{(x_1,...,x_n)} \coloneqq \mathcal{I}(\psi_{(x_1,...,x_n)}) \vee \mathcal{I}(\chi_{(x_1,...,x_n)}) \in \mathfrak{D}_{=}((C,\rho)^n)$
- $\mathcal{I}(\psi \to \chi)_{(x_1,...,x_n)} \coloneqq \mathcal{I}(\psi_{(x_1,...,x_n)}) \to \mathcal{I}(\chi_{(x_1,...,x_n)}) \in \mathfrak{D}_{=}((C,\rho)^n)$
- $\mathcal{I}(\forall x_{n+1}.\psi(x_{n+1}))_{(x_1,...,x_n)} \coloneqq \forall_{\pi_{n+1}}\mathcal{I}(\psi_{(x_1,...,x_n)}) \in \mathfrak{D}_{=}((C,\rho)^n)$
- $\mathcal{I}(\exists x_{n+1}.\psi(x_{n+1}))_{(x_1,...,x_n)} \coloneqq \exists_{\pi_{n+1}}\mathcal{I}(\psi_{(x_1,...,x_n)}) \in \mathfrak{D}_{=}((C,\rho)^n)$

As we did before, we need to verify the validity of weakening and substitution Lemma and to check that the identical assignation gives rise to the identical interpretation (so our definition of interpretation works exactly as it is supposed to).

In the interest of brevity, we consider only the intuitionistic case, as the classical case is a simple generalization of it.

Lemma 8.4 (weakening)**.** *Let* $\mathcal{L}$ *be a language for the intuitionistic predicative logic with equality and let* $\mathfrak{D}_{=}$ *be the hyperdoctrine as in Definition (8.5). Let* $t_{(x_1,...,x_n)}$ *be a term and* $\varphi_{(x_1,...,x_n)} \in Form_{\mathcal{L}}$ *a formula, with free variables among* $x_1,...x_n$ *and* $y \neq x_i$ $(i = 1,...,n)$*. Then:*

(i) $\mathcal{I}(t)_{(x_1,...x_j,y,x_{j+1},...,x_n)} = t^{\mathcal{I}}_{(x_1,...x_j,x_{j+1},...,x_n)} \circ (\pi_1 \times ... \times \pi_j \times \pi_{j+1} \times ... \times \pi_n)$

(ii) $\mathcal{I}(\varphi)_{(x_1,...x_j,y,x_{j+1},...,x_n)} = D(\pi_1 \times ... \times \pi_j \times \pi_{j+1} \times ... \times \pi_n)(\mathcal{I}(\varphi)_{(x_1,...x_j,x_{j+1},...,x_n)})$

Proof. It follows straightforwardly from Lemma (8.1) with $\mathcal{C} \equiv \mathcal{H} - \mathbf{Set}$ and from the fact that (using the same notation of Lemma (8.1)) if also $t'_{(x_1,...,x_n)}$ is a term then:

$$\begin{gathered}
\mathcal{I}((t = t')_{(-,y,-)}) = \\
= \mathfrak{D}_{=}(t^{\mathcal{I}}_{(-,y,-)} \times t'^{\mathcal{I}}_{(-,y,-)})(\rho) = \\
= \mathfrak{D}_{=}((t^{\mathcal{I}}_{(\vec{x})} \circ (\vec{\pi})) \times (t'^{\mathcal{I}}_{(\vec{x})} \circ (\vec{\pi})))(\rho) = \\
= \mathfrak{D}_{=}((t^{\mathcal{I}}_{(\vec{x})} \times t'^{\mathcal{I}}_{(\vec{x})}) \circ (\vec{\pi}))(\rho) = \\
= \mathfrak{D}_{=}(t^{\mathcal{I}}_{(\vec{x})} \times t'^{\mathcal{I}}_{(\vec{x})}) \circ \mathfrak{D}_{=}(\vec{\pi})(\rho) = \\
= \mathfrak{D}_{=}(\vec{\pi}) \circ \mathfrak{D}_{=}(t^{\mathcal{I}}_{(\vec{x})} \times t'^{\mathcal{I}}_{(\vec{x})})(\rho) = \\
= \mathfrak{D}_{=}(\vec{\pi}) \circ (\mathcal{I}(t = t')_{(\vec{x})})
\end{gathered}$$

□

[5]Indeed, by transitivity and simmetricity in Definition (8.5):

$$\rho(t,t') \wedge \rho(s,s) \wedge \rho(s',s') \leq \rho(t,s) \wedge \rho(t,s') \leq \rho(s,t) \wedge \rho(t,s') \leq \rho(s,s')$$

Lemma 8.5 (substitution). *Let $\mathcal{L}$ be a language for the intuitionistic predicative logic with equality and let $\mathfrak{D}_=$ be the hyperdoctrine as in Definition (8.5). Let $t_{(x_1,...,x_n)}$ be a term and $\varphi_{(x_1,...,x_n)} \in Form_{\mathcal{L}}$ a formula, with free variables among $x_1,...x_n$. Then:*

(i) $\mathcal{I}(t_{(x_1,...,x_n)}[s/x_i]) = t^{\mathcal{I}} \circ (\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)$

(ii) $\mathcal{I}(\varphi_{(x_1,...,x_n)}[s/x_i]) = \mathfrak{D}_=(\pi_1 \times ... \times s^{\mathcal{I}} \times ... \times \pi_n)(\mathcal{I}(\varphi)_{(x_1,...,x_n)})$

Proof. It follows straightforwardly from Lemma (8.2) with $\mathcal{C} \equiv \mathcal{H} - \mathbf{Set}$ and from the fact that (using the same notation of Lemma (8.2)) if also $t'_{(x_1,...,x_n)}$ is a term then:

$$\begin{gathered}\mathcal{I}((t = t')[s/x_i]) = \\ = \mathfrak{D}_=(t[s/x_i]^{\mathcal{I}}, t'[s/x_i]^{\mathcal{I}})(\rho) = \\ = \mathfrak{D}_=(t^{\mathcal{I}} \circ (\pi_1 \times s^{\mathcal{I}} \times \pi_2) \times t'^{\mathcal{I}} \circ (\pi_1 \times s^{\mathcal{I}} \times \pi_2))(\rho) = \\ = \mathfrak{D}_=((t^{\mathcal{I}} \times t'^{\mathcal{I}}) \circ (\pi_1 \times s^{\mathcal{I}} \times \pi_2))(\rho) = \\ = \mathfrak{D}_=(\pi_1 \times s^{\mathcal{I}} \times \pi_2) \circ \mathfrak{D}_=(t^{\mathcal{I}} \times t'^{\mathcal{I}})(\rho) = \\ = \mathfrak{D}_=(\pi_1 \times s^{\mathcal{I}} \times \pi_2)(\mathcal{I}(t = t'))\end{gathered}$$

□

Lemma 8.6. *Let $\mathcal{L}$ be a language for the intuitionistic predicative logic with equality and let $\mathfrak{D}_=$ be the hyperdoctrine as in Definition (8.5). Let's consider the assignation $c_i^{\mathcal{I}} = c_i$, $f_i^{\mathcal{I}} = f_i$, $P_j(t_1,...,t_{m_j})^{\mathcal{I}} = [P_j(t_1,...,t_{m_j})]$ and let ρ be the equivalent class of th equality of the given language. Then for every $\varphi_{(x_1,...,x_n)} \in Form_{\mathcal{L}}$ with free variables among $(x_1,...,x_n)$:*

$$\mathcal{I}(\varphi)_{(x_1,...,x_n)} = [\varphi_{(x_1,...,x_n)}]$$

where $[\varphi_{(x_1,...,x_n)}]$ is the equivalent class of the proposition φ in the Lindenbaum algebra $\mathcal{A}(\mathcal{L})$ based on the language $\mathcal{L}$.

Proof. The proof follows exactly the same steps of the proof of Theorem (8.3). □

8.4 Theorem of Soundness and Completeness for predicate logic with equality

Theorem 8.3 (Soundness). *Let $\mathcal{L}$ be a language for the (either intuitionistic or classical) predicative logic and let $Form_{\mathcal{L}}$ be the set of its formulas. Let $\Gamma_{(x_1,...,x_n)}, \varphi_{(x_1,...,x_n)} \in Form_{\mathcal{L}}$. Let $\mathcal{H}$ be a complete (either intuitionistic or classical) algebra and $(C,\rho) \in Ob(\mathcal{H} - \mathbf{Set})$ be fixed. If $\Gamma_{(x_1,...,x_n)} \vdash \varphi_{(x_1,...,x_n)}$, then for every hyperdoctrine $\mathfrak{D}_=$ of the form of Definition (8.5) and assignation $c_i^{\mathcal{I}}$, $f_i^{\mathcal{I}}$, $P_j^{\mathcal{I}}$: $\mathcal{I}_{\mathfrak{D}_=}(\Gamma) \le \mathcal{I}_{\mathfrak{D}_=}(\varphi)$ in $\mathfrak{D}_=((C,\rho)^n)$.*

Proof. The proof follows exactly the same strategy of the proof of Theorem (8.1), hence it is by induction on the complexity of the derivation tree π of the sequent $\Gamma \vdash \varphi$ (we will use the same abbreviations used on that occasion). Most cases were already proved in the proof of Theorem (8.1), there are only three cases left:

- The sequent $\Gamma \vdash \varphi$ is the axiom $= -ax$, thus $\varphi \equiv t = t$ for some term t. Then:

$$\mathcal{I}_{\mathfrak{D}_=}(\Gamma) \le \mathcal{I}_{\mathfrak{D}_=}(t = t) = \mathcal{I}_{\mathfrak{D}_=}(\varphi)$$

  since by Definition (8.6) and by reflexivity in Definition (8.5):

$$\mathcal{I}_{\mathfrak{D}_=}(t = t) = \mathfrak{D}_=(t^{\mathcal{I}} \times t^{\mathcal{I}})(\rho) = \top$$

  and trivially $\mathcal{I}_{\mathfrak{D}_=}(\Gamma) \le \top$ for any Γ.

- Let's suppose that the last step of π is the application of the $=_1$ rule, in case $\Gamma \equiv \Sigma, \Gamma'(s), t = s$ and $\varphi \equiv \varphi(s)$:

$$\dfrac{\begin{array}{c}\vdots\ \pi'\\ \Sigma, t = s, \Gamma'(t) \vdash \varphi(t)\end{array}}{\Sigma, \Gamma'(s), t = s \vdash \varphi(s)} =_1$$

By induction hypothesis we have that:

$$\mathcal{I}_{\mathfrak{D}_=}(\Sigma \wedge t = s \wedge \Gamma'(s)) \leq \mathcal{I}_{\mathfrak{D}_=}(\varphi(s))$$

However, owing to substitutivity in Definition (8.3), one infers that:

$$\mathcal{I}_{\mathfrak{D}_=}(\Sigma \wedge \Gamma'(s) \wedge t = s) \leq \mathcal{I}_{\mathfrak{D}_=}(\Sigma \wedge \Gamma'(t))$$

and

$$\mathcal{I}_{\mathfrak{D}_=}(\varphi(t) \wedge t = s) \leq \mathcal{I}_{\mathfrak{D}_=}(\varphi(s))$$

hence: $\mathcal{I}_{\mathfrak{D}_=}(\Sigma \wedge \Gamma'(s) \wedge t = s) \leq \mathcal{I}_{\mathfrak{D}_=}(\varphi(s))$.

- Let's suppose that the last step of π is the application of the $=_2$ rule, in case $\Gamma \equiv \Sigma, \Gamma'(s), s = t$ and $\varphi \equiv \varphi(s)$:

$$\dfrac{\begin{array}{c}\vdots\ \pi'\\ \Sigma, s = t, \Gamma'(t) \vdash \varphi(t)\end{array}}{\Sigma, \Gamma'(s), s = t \vdash \varphi(s)} =_2$$

We proceed in the same way as we did for the foregoing case.

□

Theorem 8.4 (Completeness)**.** *Let $\mathcal{L}$ be a language for the (either intuitionistic or classical) predicative logic and let $Form_{\mathcal{L}}$ be the set of its formulas. Let $\Gamma_{(x_1,\dots,x_n)}, \varphi_{(x_1,\dots,x_n)} \in Form_{\mathcal{L}}$. If $\mathcal{I}_{\mathfrak{D}_=}(\Gamma) \leq \mathcal{I}_{\mathfrak{D}_=}(\varphi)$ in $\mathfrak{D}_=(C^n)$ for every hyperdoctrine $\mathfrak{D}_=$ and assignation $c_i^{\mathcal{I}}$, $f_i^{\mathcal{I}}$, $P_j^{\mathcal{I}}$, then: $\Gamma_{(x_1,\dots,x_n)} \vdash \varphi_{(x_1,\dots,x_n)}$.*

Proof. The proof follows exactly the same strategy of the proof of Theorem (8.2), although using Lemma (8.6) instead of Lemma (8.3). □

8.5 An example

We want to give an example of the use of the notion of interpretation in Definition (8.6) in order to produce a countermodel for the formula $\forall x_1 \forall x_2.(x_1 = x_2 \vee x_1 \neq x_2)$, which we know not to be intuitionistically valid. In other words, we want to show that

$$\nvdash_{HA} \forall x_1 \forall x_2.(x_1 = x_2 \vee \neg x_1 = x_2)$$

by exhibiting an interpretation that falsifies this sentence.

First of all, we notice that an interpretation as presented in Definition (8.6) is given by the choice of a Heyting algebra $\mathcal{H}$, the choice of an object $(C, \rho) \in Ob(\mathcal{H} - \mathbf{Set}^{op})$ (which means fixing a set C and an $\mathcal{H}$-valued equivalence relation ρ) and the choice of a hyperdoctrine $\mathfrak{D}_=$ of the form

$$\mathfrak{D}_= : (\mathcal{H} - \mathbf{Set})^{op} \longrightarrow \mathbf{Ha}$$

As far as our initial purpose is concerned, we make the following choices. We consider the Heyting algebra

$$\mathcal{H} \equiv \tau_{\mathbb{R}}$$

given by the real numbers with the canonical topology, i.e. a subset $X \subseteq \mathbb{R}$ is open if and only if it is union of open intervals (we recall that topological spaces are examples of complete Heyting algebras). Let $C \in Ob(\mathbf{Set})$ be a set with two elements

$$C \equiv \{c, c'\}$$

and let's define a $\tau_{\mathbb{R}}$-valued equivalence relation ρ on C by:

$$\begin{array}{rccc} \rho: & C \times C & \longrightarrow & \tau_{\mathbb{R}} \\ & (c_1, c_1) & \longmapsto & \{1\} \\ & (c_2, c_2) & \longmapsto & \{1\} \\ & (c_1, c_2) & \longmapsto &]0,1[\\ & (c_2, c_1) & \longmapsto &]0,1[\end{array}$$

We consider the hyperdoctrine for predicative intuitionistic logic with equality $IL_=$:

$$\mathfrak{D}_= : (\tau_{\mathbb{R}} - \mathbf{Set})^{op} \longrightarrow \mathbf{Ha}$$

where for every object $(C,\rho) \in Ob((\tau_{\mathbb{R}} - \mathbf{Set})^{op})$: $\delta_C \equiv \rho$ and if $\pi_i : (C,\rho) \times (C,\rho) \longrightarrow (C,\rho)$ and z is the variable such that $\mathcal{I}(z) = \pi_i$ then $\exists_{\pi_i} \equiv \bigcup_{[c/z], c \in C}$ $\forall_{\pi_i} \equiv \bigcap_{[c/z], c \in C}$.

By definition of interpretation of terms we gather:

$$x_1^{\mathcal{I}} = \pi_1 : (C,\rho)^2 \longrightarrow (C,\rho) \text{ and } x_2^{\mathcal{I}} = \pi_2 : (C,\rho)^2 \longrightarrow (C,\rho)$$

and by definition of interpretation of formulas we can deduce:

$$\mathcal{I}((x_1 = x_2)_{(x_1,x_2)}) = \mathfrak{D}_=(x_1^{\mathcal{I}} \times x_2^{\mathcal{I}})(\rho) = \rho \circ (\pi_1, \pi_2)$$

and

$$\mathcal{I}((\neg x_1 = x_2)_{(x_1,x_2)}) = \neg_{\tau_{\mathbb{R}}} \mathcal{I}((x_1 = x_2)_{(x_1,x_2)}) = \neg_{\tau_{\mathbb{R}}}(\rho \circ (\pi_1,\pi_2)) = (\rho \circ (\pi_1,\pi_2))^c$$

in particular:

$$\mathcal{I}((x_1 = x_2)_{(x_1,x_2)})(c_1,c_2) = (\rho \circ (\pi_1,\pi_2))(c_2,c_1) =]0,1[$$
$$\mathcal{I}((x_1 = x_2)_{(x_1,x_2)})(c_1,c_1) = (\rho \circ (\pi_1,\pi_2))(c_2,c_2) = \{1\}$$

and

$$\mathcal{I}((\neg x_1 = x_2)_{(x_1,x_2)})(c_1,c_2) = \mathcal{I}((\neg x_1 = x_2)_{(x_1,x_2)})(c_2,c_1) = \mathbb{R} \setminus]0,1[$$
$$\mathcal{I}((\neg x_1 = x_2)_{(x_1,x_2)})(c_1,c_1) = \mathcal{I}((\neg x_1 = x_2)_{(x_1,x_2)})(c_2,c_2) = \mathbb{R} \setminus \{1\}$$

By the inductive Definition (8.6) of interpretation we conclude that:

$$\mathcal{I}((x_1 = x_2 \vee \neg x_1 = x_2)_{(x_1,x_2)}) = \mathcal{I}((x_1 = x_2)_{(x_1,x_2)}) \vee_{\tau_{\mathbb{R}}} \mathcal{I}((\neg x_1 = x_2)_{(x_1,x_2)}) =$$
$$\mathcal{I}((x_1 = x_2)_{(x_1,x_2)}) \cup \mathcal{I}((\neg x_1 = x_2)_{(x_1,x_2)})$$

in particular:

$$\mathcal{I}((x_1 = x_2 \vee \neg x_1 = x_2)_{(x_1,x_2)})(c_1,c_2) = \mathcal{I}((x_1 = x_2 \vee \neg x_1 = x_2)_{(x_1,x_2)})(c_2,c_1)]0,1[\cup(\mathbb{R} \setminus]0,1[) = \mathbb{R} \setminus \{0,1\}$$
$$\mathcal{I}((x_1 = x_2 \vee \neg x_1 = x_2)_{(x_1,x_2)})(c_1,c_1) = \mathcal{I}((x_1 = x_2 \vee \neg x_1 = x_2)_{(x_1,x_2)})(c_2,c_2)\{1\} \cup (\mathbb{R} \setminus \{1\}) = \mathbb{R}$$

And finally, by Definition (8.6) of interpretation we find:

$$\begin{gathered} \mathcal{I}(\forall x_1 \forall x_2.(x_1 = x_2 \vee \neg x_1 = x_2)) = \\ = \forall_{\pi_1} \mathcal{I}((\forall x_2.(x_1 = x_2 \vee \neg x_1 = x_2))_{(x_1)}) = \\ = \forall_{\pi_2} \forall_{\pi_1} \mathcal{I}(x_1 = x_2 \vee \neg x_1 = x_2) = \\ = \bigcap_{[c/x_2], c \in C} \bigcap_{[c'/x_1], c' \in C} \mathcal{I}(x_1 = x_2 \vee \neg x_1 = x_2) = \\ = \mathcal{I}(x_1 = x_2 \vee \neg x_1 = x_2)(c_1,c_1) \cap \mathcal{I}(x_1 = x_2 \vee \neg x_1 = x_2)(c_1,c_2) \cap \\ \cap \mathcal{I}(x_1 = x_2 \vee \neg x_1 = x_2)(c_2,c_1) \cap \mathcal{I}(x_1 = x_2 \vee \neg x_1 = x_2)(c_2,c_2) = \\ \mathbb{R} \cap \mathbb{R} \cap \mathbb{R} \setminus \{0,1\} \cap \mathbb{R} \setminus \{0,1\} = \mathbb{R} \setminus \{0,1\} \end{gathered}$$

thus the formula $\forall x_1 \forall x_2.(x_1 = x_2 \vee x_1 \neq x_2)$ is not intuitionistically valid since there is a model in which it is not satisfied.

8.6 A global view

Looking at the mathematical and historical development the first period of the evolution of categorical logic, which we have tried so far to depict in its most important implications, makes us understand at least two crucial reasons why categorical logic gained broad approval among mathematicians. Firstly, a translation in category-theoretic terms of the key concepts of logic and set theory is fruitful, provided they fall under the central notions of category theory, namely adjoint functors. Secondly, the proper translation revealed how many notions of algebraic geometry and algebraic topology were mathematically equivalent to logical notions. This latter aspect is crucial: the categorical analysis of logical notions was at first associated with a distinctive ideological component, on some occasions strongly political, according to which in 1960s and 1970s logic and geometry were opposites, however soon the purely algebraic fabric weaving together all these aspects was revealed and all parties agreed on the fact that this opened new avenues of research and offered new ways of thinking.

Of course, we need to underline how these observations regard mainly the mathematical world. What is curious about categorical logic is that complaints and objections to the use of this new toolbox came from the logicians' world itself. The doubts raised by mathematical logicians towards categorical logicians can be summarized by the following words of Marquis and Reyes:

> "[...] can you prove something by these [i.e. categorical] means that cannot be proved by other, meaning more traditional, means? If what one does in categorical logic is simply the same but in a different, more complicated, and ultimately irrelevant guise, why bother learning this general abstract nonsense in the first place? If one does not care about a grand unifying picture of mathematics and its foundations, of unsuspected links between domains resulting in a complex and rich network of abstract structures, then is there any genuine value in the categorical toolbox?"([MR11], pg 105)

Many categorical logicians spent a hard time trying to convince mathematical logicians that their analyses were indeed foundational. The point is deeply ideological or, better to say, philosophical: the algebraic approach to foundations has been seen since Frege misguided because it implies viewing logic as a part of mathematics, while Frege and his successors assumed exactly the opposite. Nevertheless, we should recognize that categorical logic does in fact offer a genuine foundational analysis of many mathematical concepts, and in addition to that it offers the great advantage of being a bridge between constructive and classical approaches, geometry and logic, topology and logic, theoretical computer science and logic, just to name a few of them.

Besides these observations, it should be unanimously clear why categorical logic had also philosophical consequences. The main field of the philosophy of mathematics that categorical logic affected was the so called **mathematical (or logical) structuralism**. According to it, mathematics studies structures and mathematical objects are nothing but positions in those structures, so mathematical theories consist in the description of structures of mathematical objects. As a consequence, mathematical objects do not own any intrinsic property, but rather they are in external relations one to each other within a given system. On the logical side, structuralists believe that mathematical statements have an objective truth value, thus their epistemological views are totally realistic. Ontologically speaking, though, they are not interested in the type of existence that mathematical objects and structures have: their focus is on what kind of entity a mathematical object can be ascribed to (of course, this does depend on the kind of structure which a particular object belongs to). The main contributions as regards structuralism in the philosophy of mathematics came from Paul Benacerraf (1931), Michael Resnik (March 20, 1938) and Stewart Shapiro (1951).

Although the main definition of structuralism is well-accepted, it can be elaborated in various different manners and, perhaps also for this reason, not all categorical theorists have declared themselves structuralists. Indeed, we can interpret structuralism in two different ways and we can distinct two possible levels at which it can be developed. First of all, claiming that mathematics is the study of structures can mean either that mathematics is *about structures* or that mathematics is *about structured systems*, i.e. systems that possess a structure. Different investigations will of course lead to different outcomes. In addition to that, we can find both a concrete and an abstract level inside the development of structuralism.

At the **concrete level** mathematical structuralism is also called **model-structuralism** and is the philosophical claim that the subject matter of a particular mathematical theory is concrete kinds of structured systems (models) and their morphology. Hence, a given mathematical object turns out to be a position in a concrete system, which has a specific structure. The role of a mathematical theory is characterizing such kinds of objects in terms of the common structure of the concrete systems in which they are positions (this means characterizing them *up to isomorphism*). Let's take for example the theory of natural numbers **PA** given by Peano axioms. This provides a framework to present the models that have the same natural-number structure and that are isomorphic one to each other. In this respect, natural numbers are positions in any interpretation that satisfies Peano axioms.

At the **abstract level**, mathematical structuralism is the philosophical claim that the subject matter of mathematics itself is abstract kinds of structured systems and their morphology. In this case, a mathematical object is nothing but a position in an abstract system, which itself has an abstract structure. Such types of abstract systems are described in terms of their common structures by mathematical theories. Whereas in the case of concrete structure it is clear that we are dealing with models, it not unanimously clear what an abstract structure is. In order to respond to such a question, we have to distinguish between two possible interpretations and three varieties of philosophically positioned structuralism. The abstract structuralism can be interpreted as:

- **realist structuralism** (or **ante rem structuralism**): structures exist as objects of study in their own right, so any structure exists regardless of any system exemplifying it.

- **nominalist structuralism** (or **ante rem structuralism**): mathematical objects and structures do not exist by themselves, rather structures are systems of objects behaving in a certain manner and speaking about a particular structure is allowed after having provided a generalization over all systems of a certain type.

This distinction recalls that one between Frege and Hilbert's approach at the concrete level. An ante re structuralist sees abstract structures as anything that satisfies the axioms that are taken to characterize the abstract type of structure under consideration. An in re structuralist eschews the Frege demand that before talking about abstract structured systems as object, that is positions in a type of structured system, there must be a background theory that allows dealing with typed of structures as independently existing things. Depending on the way according to which preconditions for the independent existence of abstract structures, required by Frege demand, are settled, we single out three varieties of abstract structuralism. These suggest set theory, structure theory or modal logic as tools to give conditions to assert whether a given abstract system have or does not have a structure of a specific type.

Categorical logic fits in among all these interpretations and varieties of mathematical structuralism in many, sometimes complicated, ways. The standard assertion is that categories are types of structured sets and in this respect category theory can be taken as a framework to develop a particular variety of abstract structuralism. Nevertheless, it seems that this claim may not "do justice to the actual practise of category theory and [...] fails to recognize the fact that category theory is both a foundational and philosophical alternative" (Landry, E. and Marquis, J. P., *Categories in context: Historical, Foundational, and Philosophical*, Philosophia Mathematica, 1, 13, 2005 , pg 29). Unfortunately, such an argumentation, interesting and worthy though it might be, would require a great command in the subject and is beyond the scope of the thesis.

Chapter 9

Kleene's intuitionistic realizability interpretation as hyperdoctrine

This chapter offers another example in which categorical language can be used to formalize logical notions.
In the first part of the Chapter we introduce some basic notions of recursion theory, which are essential to deal with BHK-interpretation and Kleene's realizers. After introducing Kleene's realizability theory in the canonical way, we will show a way to treat it categorically, namely we will present the hyperdoctrine of realizability. Furthermore, we will use adjunctions in order to show the well-known result claiming that the rules of intuitionistic predicate logic IL are valid in Kleene's realizability whilst those of classical predicate logic CL are not.

9.1 Classical and intuitionistic Peano arithmetics PA and HA

Let's consider the classical and intuitionistic calculus with equality $\mathbf{CL}_=$ and $\mathbf{IL}_=$ defined as follows.

Definition 9.1. *Let $\mathbf{CL}_=$ be the calculus of the classical logic with identity, containing the following rules*[1]*, where φ, ψ are arbitrary propositions:*

$$\overset{id-ax}{\Gamma,\varphi,\Gamma' \vdash \Delta,\varphi,\Delta'} \qquad \overset{\bot-ax}{\Gamma,\bot,\Gamma' \vdash \Delta} \qquad \overset{\top-ax}{\Gamma \vdash \Delta,\top,\Delta'} \qquad \overset{=-ax}{\Gamma \vdash t=t,\Delta}$$

$$\frac{\Gamma,\varphi,\psi \vdash \Delta}{\Gamma,\varphi\wedge\psi \vdash \Delta}\ \wedge l \qquad \frac{\Gamma \vdash \varphi,\Delta \quad \Gamma \vdash \psi,\Delta}{\Gamma \vdash \varphi\wedge\psi,\Delta}\ \wedge r$$

$$\frac{\Gamma,\varphi \vdash \Delta \quad \Gamma,\psi \vdash \Delta}{\Gamma,\varphi\vee\psi \vdash \Delta}\ \vee l \qquad \frac{\Gamma \vdash \varphi,\psi,\Delta}{\Gamma \vdash \varphi\vee\psi,\Delta}\ \vee r$$

$$\frac{\Gamma \vdash \varphi,\Delta}{\Gamma,\neg\varphi \vdash \Delta}\ \neg l \qquad \frac{\Gamma,\varphi \vdash \Delta}{\Gamma \vdash \neg\varphi,\Delta}\ \neg r$$

$$\frac{\Gamma \vdash \varphi,\Delta \quad \Gamma,\psi \vdash \Delta}{\Gamma,\varphi\to\psi \vdash \Delta}\ \to l \qquad \frac{\Gamma,\varphi \vdash \psi,\Delta}{\Gamma \vdash \varphi\to\psi,\Delta}\ \to r$$

[1] We implicitly allow the possibility of swapping ad lib formulas or collections of formulas in a sequent, paying attention not to make an antecedent a consequent or the other way round. More precisely, we imply the following exchange rules:

$$\frac{\Sigma,\Gamma,\Theta,\Gamma',\Delta \vdash \Sigma'}{\Sigma,\Gamma',\Theta,\Gamma,\Delta \vdash \Sigma'}\ ex_l \qquad \frac{\Gamma \vdash \Sigma,\Delta,\Theta,\Delta',\Gamma'}{\Gamma \vdash \Sigma,\Delta',\Theta,\Delta,\Gamma'}\ ex_r$$

$$\frac{\Gamma, \forall x.\varphi(x), \varphi(t) \vdash \Delta}{\Gamma, \forall x.\varphi(x) \vdash \Delta}\ \forall l \qquad \frac{\Gamma \vdash \varphi(w), \Delta}{\Gamma \vdash \forall x.\varphi(x), \Delta}\ \forall r (w \notin FV(\Gamma, \forall x.\varphi(x), \Delta))$$

$$\frac{\Gamma, \varphi(w) \vdash \Delta}{\Gamma, \exists x.\varphi(x) \vdash \Delta}\ \exists l (w \notin FV(\Gamma, \exists x.\varphi(x), \Delta)) \qquad \frac{\Gamma \vdash \varphi(t), \exists x.\varphi(x), \Delta}{\Gamma \vdash \exists x.\varphi(x), \Delta}\ \exists r$$

$$\frac{\Sigma, t = s, \Gamma(t) \vdash \Delta(t), \Theta}{\Sigma, \Gamma(s), t = s \vdash \Delta(s), \Theta} =_1 \qquad \frac{\Sigma, s = t, \Gamma(t) \vdash \Delta(t), \Theta}{\Sigma, \Gamma(s), s = t \vdash \Delta(s), \Theta} =_2$$

Definition 9.2. *Let* $\mathbf{IL}_=$ *be the calculus of intuitionistic logic with identity, containing the following rules*[2], *where* φ, ψ *and* $\mathfrak{pr}$ *are arbitrary propositions:*

$$\overset{id-ax}{\Gamma, \varphi, \Gamma' \vdash \varphi} \qquad \overset{\perp-ax}{\Gamma, \perp, \Gamma' \vdash \mathfrak{pr}} \qquad \overset{\top-ax}{\Gamma \vdash \top} \qquad \overset{=-ax}{\Gamma \vdash t = t}$$

$$\frac{\Gamma, \varphi, \psi \vdash \mathfrak{pr}}{\Gamma, \varphi \wedge \psi \vdash \mathfrak{pr}}\ \wedge l \qquad \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}\ \wedge r$$

$$\frac{\Gamma, \varphi \vdash \mathfrak{pr} \qquad \Gamma, \psi \vdash \mathfrak{pr}}{\Gamma, \varphi \vee \psi \vdash \mathfrak{pr}}\ \vee l \qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ \vee r_1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ \vee r_2$$

$$\frac{\Gamma, \neg\varphi \vdash \varphi}{\Gamma, \neg\varphi \vdash \mathfrak{pr}}\ \neg l \qquad \frac{\Gamma, \varphi \vdash}{\Gamma \vdash \neg\varphi}\ \neg r$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma, \psi \vdash \mathfrak{pr}}{\Gamma, \varphi \to \psi \vdash \mathfrak{pr}} \to l \qquad \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \to r$$

$$\frac{\Gamma, \forall x.\varphi(x), \varphi(t) \vdash \mathfrak{pr}}{\Gamma, \forall x.\varphi(x) \vdash \mathfrak{pr}}\ \forall l \qquad \frac{\Gamma \vdash \varphi(w)}{\Gamma \vdash \forall x.\varphi(x)}\ \forall r (w \notin FV(\Gamma, \forall x.\varphi(x)))$$

$$\frac{\Gamma, \varphi(w) \vdash \mathfrak{pr}}{\Gamma, \exists x.\varphi(x) \vdash \mathfrak{pr}}\ \exists l (w \notin FV(\Gamma, \exists x.\varphi(x), \mathfrak{pr})) \qquad \frac{\Gamma \vdash \varphi(t)}{\Gamma \vdash \exists x.\varphi(x)}\ \exists r$$

$$\frac{\Sigma, t = s, \Gamma(t) \vdash \varphi(t)}{\Sigma, \Gamma(s), t = s \vdash \varphi(s)} =_1 \qquad \frac{\Sigma, s = t, \Gamma(t) \vdash \varphi(t)}{\Sigma, \Gamma(s), s = t \vdash \varphi(s)} =_2$$

We introduce now the classical and intuitionistic Peano arithmetics **PA** and **HA** as particular logical theories.

Definition 9.3. *A classical [intuitionistic]* ***theory*** *is obtained by extending the calculus of classical [intuitionistic] logic with identity* $\mathbf{CL}_=$ *[*$\mathbf{IL}_=$*] with some extralogic axioms and with the right and left composition rules:*

$$\frac{\Gamma' \vdash \mathfrak{fr} \qquad \Gamma, \mathfrak{fr}, \Gamma'' \vdash \Delta}{\Gamma, \Gamma', \Gamma'' \vdash \Delta}\ l - comp \qquad \frac{\Gamma \vdash \Delta, \mathfrak{fr}, \Delta'' \qquad \mathfrak{fr} \vdash \Delta'}{\Gamma \vdash \Delta, \Delta', \Delta''}\ r - comp$$

Let's consider a predicative language for the classical logic, endowed with a constant 0 and the symbols s, $+$ and $\cdot$ for the successor, the sum and the product.

Definition 9.4. *The classical theory of Peano arithmetic* **PA** *is given by the rules of* $\mathbf{CL}_=$, *the rules of left and right composition* $l - comp$ *and* $r - comp$, *and the following axioms:*

$$Ax1 \vdash \forall x.s(x) \neq 0$$

$$Ax2 \vdash \forall x \forall y.(s(x) = s(y) \to x = y)$$

[2]As before, we implicitly admit the exchange rule.

$$Ax3 \vdash \forall x.x + 0 = x$$

$$Ax4 \vdash \forall x \forall y.(x + s(y) = s(x + y)$$

$$Ax5 \vdash \forall x.x \cdot 0 = 0$$

$$Ax6 \vdash \forall x \forall y.(x \cdot s(y) 0 x \cdot y + x$$

$$Ax7 \vdash (\mathfrak{fr} \wedge \forall x.(\mathfrak{fr}(x) \to \mathfrak{fr}(s(x))) \to \forall x.\mathfrak{fr}(x)$$

Definition 9.5. *The intuitionistic theory of Peano arithmetic* **HA** *is given by the rules of* $\mathbf{IL}_=$*, the rules of left and right composition* $l - comp$ *and* $r - comp$*, and the same list of axioms of* **PA**.

Equivalently, we could define **PA** as the extension of **HA** obtained by adding either the Law of Excluded Middle $\varphi \vee \neg\varphi$ or the Double Negation $\neg\neg\varphi \to \varphi$ to its axioms. Both in **PA** and **HA**, the numeral n is given by the application of the successor s to 0 a number n of times.

9.2 BHK-interpretation and Kleene's realizability

Kleene's realizability was first introduced by Kleene in the 1940s with the specific aim of giving to BHK-interpretation a precise meaning. After its birth, the theory of realizability went through many changes and reformulations. One of the most successful was Kreisel's modified realizability (1959), according to which to every formula φ, say in the language of **HA**, a new formula $x\mathbf{mr}\varphi$ (to be read "x modified realizes φ") always in the language of **HA** is assigned. Modifying in a proper way Kreisel's modified realizability itself, one can obtain the so-called term extraction for **HA** and even for Π^0_2-formulas of **PA**.

The BHK-interpretation (with B standing for Brouwer, H for Heyting and K for Kolmogorov) is a description of proofs of propositions as (computable) functions. This is otherwise known as the paradigm of propositions as types and proofs as programs. The name "Brouwer-Heyting-Kolmogorov" is due to Troelstra, but not everybody agrees with this notation for some controversial historical reasons: Brouwer never explicitly formulated any interpretation of this sort, and remained against all formalism his entire life, therefore it might be better to use the name "Heyting-Kolmogorov". The crucial idea behind the BHK-interpretation is that the meaning of a formula should not be explained in terms of what makes it true, instead it should be explained in terms of what counts as a proof of that formula.

Definition 9.6. *A proof* $\mathbf{p}$ *of a formula is defined inductively on the complexity of the formula as follows:*

- *there is no proof for* $\bot$*;*
- *a proof* $\mathbf{p}$ *of* $\varphi \wedge \psi$ *consists of a pair* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)$ *where* $\mathbf{p}_1$ *is a proof of* φ *and* $\mathbf{p}_2$ *is a proof of* ψ*;*
- *a proof* $\mathbf{p}$ *of* $\varphi \vee \psi$ *consists of a pair* $\mathbf{p} = (k, \mathbf{p}')$ *where* $k = 0$ *and* $\mathbf{p}'$ *is a proof of* φ *or* $k = 1$ *and* $\mathbf{p}'$ *is a proof of* ψ*;*
- *a proof* $\mathbf{p}$ *of* $\varphi \to \psi$ *is an effective method (a functional construction) for transforming every proof* $\mathbf{z}$ *of* φ *into a proof* $\mathbf{p}(\mathbf{z})$ *of* ψ*;*
- *a proof* $\mathbf{p}$ *of* $\exists x.\varphi$ *consists of a pair* $\mathbf{p} = (k, \mathbf{p}')$ *where* k *is a natural number and* $\mathbf{p}'$ *is a proof of* $\varphi[k/x]$*;*
- *a proof* $\mathbf{p}$ *of* $\forall x.\varphi$ *is an effective method (a functionnal construction) for transforming every natural number* c *into a proof* $\mathbf{p}(c)$ *of* $\varphi[c/x]$.

Inside Peano's (as well as Heyting's) arithmetic it is possible to represent all partial recursive functions $f : \mathbb{N} \longrightarrow \mathbb{N}$, using predicates $\mathbf{R}(n_1, ..., n_m, k)$ such that:

$$f(n_1, ..., n_m) = k \Leftrightarrow \vdash_{\mathbf{PA}} R(n_1, ..., n_m, k)$$

$$f(n_1, ..., n_m) \neq k \Leftrightarrow \vdash_{\mathbf{PA}} \neg R(n_1, ..., n_m, k)$$

$$\vdash_{\mathbf{PA}} \forall x_1 ... \forall x_m \forall y \forall z.(R(x_1, ..., x_m, y) \wedge R(x_1, ..., x_m, z) \rightarrow y = z)$$

For convenience, we allow ourselves to write primitive recursive functions as terms, so we endow the language of **PA** and **HA** with terms like $f(x_1, ..., x_m)$ for every m-ary primitive recursive function. Also, it can be proved that there is a bijection $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$ given by a function called **surjective pairing**:

$$\begin{array}{rccc} \langle -, - \rangle : & \mathbb{N} \times \mathbb{N} & \longrightarrow & \mathbb{N} \\ & (n_1, n_2) & \longmapsto & \frac{1}{2}(n_1 + n_2)(n_1 + n_2 + 1) + n_2 \end{array}$$

whose inverse in given by the projections:

$$\begin{array}{rccc} (\pi_1, \pi_2) : & \mathbb{N} & \longrightarrow & \mathbb{N} \times \mathbb{N} \\ & n & \longmapsto & (\pi_1(n), \pi_2(n)) \end{array}$$

hence it holds that $\langle \pi_1(n), \pi_2(n) \rangle = n$.

Every natural number can be viewed with two different meanings:

- it can be an input or an output of a function, hence of a program;
- it can indicate, by Gödel's coding, the primitive recursive function $\{n\} : \mathbb{N} \to \mathbb{N}$ whose input is given by $(m_1, ..., m_k)$ and whose output is denoted by $\{n\}(m_1, ..., m_k)$,

If t is a term in the language of **PA** or **HA** with free variables $x_1, ..., x_k$, we denote by

$$\Lambda x_1, ..., x_k.t$$

the natural number standing, according to Gödel's coding, for the primitive recursive function such that, given $n_1, ..., n_k$ natural numbers, $\vdash_{\mathbf{PA}, \mathbf{HA}} \{\Lambda x_1, ..., x_k.t\}(n_1, ..., n_m) = t[n_1/x_1, ..., n_k/x_k]$. Also, if e is the coding for a program, we write $\{e\}(x)$ to mean that the program of code e converges in x and has $\{e\}(x)$ as output.

Theorem 9.1 (Kleene's Normal Form). *There exists a primitive recursive function U and for every $n > 0$ there exists a primitive recursive predicates T_n such that for all $x_1, ..., x_n$:*

$$\{e\}(x_1, ..., x_n) = U(\mu y.T_n(e, x_1, ..., x_n, y))$$

Therefore, we can introduce a primitive recursive predicate T_n (which we will simply write T), called the **Kleene's predicate**, such that, for every $e, x_1, ..., x_n, y$ natural numbers, $T(e, x_1, .., x_n, y)$ claims that there exists a successful computation y for the coded program e when computed with input $x_1, ..., x_n$. U is a primitive recursive function which gives for every successful computation y the output $U(y)$.

Proposition 9.1 (Padding Lemma). *Every partial computable function has infinitely many indices. In particular, there are primitive recursive functions f and g such that for all x and y: $f(y) > y$, $g(x, y) > x$ and $\{y\} = \{f(y)\} = \{g(x, y)\}$.*

As a consequence, there are infinitely many programs that require the same inputs and produce the same outputs, even though they are different since the instructions they contain are not the same. Even, it is really easy to give an example: it suffices to add meaningless instructions to a program.

Theorem 9.2 (s-m-n Kleene's Theorem). *For every $m, n \in \mathbb{N}$ there exists an injective primitive recursive $(n+1)$-ary function s_n^m such that for all $e, x_1, ..., x_n$, given the natural numbers $y_1, .., y_m$:*

$$\{s_n^m(e, x_1, ..., x_n)\}(y_1, ..., y_m) = \{e\}(x_1, ..., x_n, y_1, ..., y_m)$$

Corollary 9.1. *Let α be an $(m+n)$-ary partial computable function. Then there exists an injective m-ary computable function f such that for all $y_1, ..., y_m, x_1, ..., x_n$: $\alpha(y_1, ..., y_m, x_1, ..., x_m) = \{f(y_1, ..., y_m)\}(x_1, ..., x_n)$.*

We introduce now the so-called Kleene's realizability. The general idea under the realizability is that we can award to every "valid" formula a proof of its "validity", which we will refer to as the realizer of that formula. We would also like to be able to manipulate realizers in the same way we can manipulate formulas.

Definition 9.7. *Let $\mathcal{L}$ be a language for Peano's (or Heyting's) arithmetic. We define a relation $\Vdash$ between natural numbers $\mathbb{N}$ and formulas in $Form_{\mathcal{L}}$ by induction on the complexity of formulas as follows:*

- $n \Vdash \bot \equiv \bot$ *(there is no realizer for $\bot$);*
- $n \Vdash \top \equiv \top$ *($\top$ is always realized);*
- $n \Vdash \varphi \wedge \psi \equiv (\pi_1(n) \Vdash \varphi \wedge \pi_2(n) \Vdash \psi)$;
- $n \Vdash \varphi \vee \psi \equiv ((\pi_1(n) = 0 \wedge \pi_2(n) \Vdash \varphi) \vee (\pi_1(n) = 1 \wedge \pi_2(n) \Vdash \psi)$;
- $n \Vdash \varphi \rightarrow \psi \equiv \forall k.(k \Vdash \varphi \rightarrow \{n\}(k) \Vdash \psi)$;
- $n \Vdash \exists x.\varphi \equiv \pi_2(n) \Vdash \varphi[\pi_1(n)/x]$;
- $m \Vdash \forall x.\varphi \equiv \forall k.(\{n\}(k) \Vdash \varphi[k/x])$;
- $n \Vdash t = s \equiv (n = 0 \wedge t = s)$.

Since we can define the negation $\neg$ as $\neg\varphi \equiv \varphi \rightarrow \bot$, we have:

- $n \Vdash \neg\varphi \equiv n \Vdash \varphi \implies \bot \equiv \forall k.(k \Vdash \varphi \rightarrow \{n\}(k) \Vdash \bot)$.

*If $n \Vdash \varphi$ we say that n is a **realizer** and that it realizes the formula φ.*

We adopt the harpoon-vector notation $\vec{x}$ to denote a list of variables $x_1, ..., x_n$. Therefore using $\forall\vec{x}.\varphi(x)$ we mean to say $\forall x_1, ..., x_n.\varphi(x_1, ..., x_n)$. In addition, writing $\Gamma(\vec{x})$ we mean that Γ is a list of formulas in which the free variables are to be found among $\vec{x} = (x_1, ..., x_n)$.

Definition 9.8. *In the propositional language we say that $\Gamma \vdash \Delta$ is **valid in Kleene's realizability** (or simply **realizable**) if there exists a realizer n such that $n \Vdash \Gamma \rightarrow \Delta$[3]. In the predicative language we say that $\Gamma \vdash \Delta$ is **valid in Kleene's realizability** (or simply **realizable**) if there exists a realizer n such that $n \Vdash \forall\vec{y}.(\Gamma(\vec{y}) \rightarrow \Delta(\vec{y})$[4]. A rule is **valid in Kleene's realizability** (or simply **realizable**) if assuming the premises realizable, the conclusion is realizable as well.*

Theorem 9.3. *The rules of IL are valid in Kleene's realizability.*

Theorem 9.4. *The rules of CL are not valid in Kleene's realizability.*

[3] Here by Γ we mean the conjunction of all the formulas in Γ and by Δ we mean the disjunction of all the formulas in Δ.

[4] as before, by $\Gamma(\vec{y})$ we mean the conjunction of all the formulas in it and by $\Delta(\vec{y})$ we mean the disjunction of all the formulas in it.

9.3 Realizability in categorical terms

We define now new categories, where objects are the formulas in a given language $\mathcal{L}$, id est $Ob = Form_{\mathcal{L}}$, whilst maps between them change according to the meaning we would like to give to a specific category.

First of all, let's consider a language $\mathcal{L}$ and the formulas in $Form_{\mathcal{L}}$. Given two formulas φ and ψ, we can wonder if there is a derivation from φ to ψ in sequent calculus. However, there can be many different derivations form φ to ψ, to be precise there are infinitely many ones, since useless rules can be added out of anyone's choosing (for instance you can add an invertible rule and its inverse, as many time as you like). We define in $Form_{\mathcal{L}}$ a relation $\approx$ in the following way: given $\varphi, \psi \in Form_{\mathcal{L}}$, we say $\pi \approx \pi'$ if both π and π' are derivations from φ to ψ. In fact, it is an equivalent relation, so we can consider the corresponding equivalent classes and the corresponding quotient set.

Definition 9.9. *Given a predicative language $\mathcal{L}$ for the classical logic, $Seq^{\mathcal{L}}_{CL}$ is the category whose objects are the formulas in $Form_{\mathcal{L}}$. Given $\varphi, \psi \in Ob(Seq^{\mathcal{L}}_{CL}) = Form_{\mathcal{L}}$, the maps from φ to ψ are $\{\pi | \pi \text{ is a derivation } \varphi \vdash_{CL} \psi\} / \approx$.*

From the definition we gather that the set $Seq^{\mathcal{L}}_{CL}(\varphi, \psi)$ is a singleton if and only if there is a derivation from φ to ψ.

Definition 9.10. *Given a predicative language $\mathcal{L}$ for the intuitionistic logic, $Seq^{\mathcal{L}}_{IL}$ is the category whose objects are the formulas in $Form_{\mathcal{L}}$. Given $\varphi, \psi \in Ob(Seq^{\mathcal{L}}_{IL}) = Form_{\mathcal{L}}$, the maps from φ to ψ are $\{\pi | \pi \text{ is a derivation } \varphi \vdash_{IL} \psi\} / \approx$.*

From the definition we gather that the set $Seq^{\mathcal{L}}_{IL}(\varphi, \psi)$ is a singleton if and only if there is a derivation from φ to ψ. Thanks to Gentzen Theorem about the elimination of the cut rule, $Seq^{\mathcal{L}}_{CL}$ and $Seq^{\mathcal{L}}_{IL}$ are categories.

Let's consider now two formulas $\varphi, \psi \in Form_{\mathcal{L}}$, where $\mathcal{L}$ is a language for Heyting arithmetic, and suppose that there is a realizer e such that $e \Vdash \varphi \to \psi$. A realizer is nothing but a program, and so there could be many different computations, which may a priori need different inputs in order to run, giving nevertheless the same outputs. Not all of them are realizers for $\varphi \to \psi$, though: we should ask that those programs take realizers of φ as inputs, in addition to giving the right outputs. By Padding Lemma (9.1) there are infinitely many different realizers for $\varphi \to \psi$, which are programs that need realizers of φ as inputs and produce the realizers of ψ as outputs.

We can define on the set of realizers (which is coded in fact by the set of natural numbers $\mathbb{N}$) the following relation[5]:

$$e \approx e' \overset{def}{\Leftrightarrow} \forall \vec{x} \exists y \exists y'.(T(e, \vec{x}, y) \wedge T(e', \vec{x}, y') \wedge U(y) = U(y'))$$

This is an equivalence relation on the set of all realizers: it is trivially reflexive, it is symmetric since the condition that defines $\approx$ is symmetric, and it is transitive because of the transitivity of equality. Hence we can take into consideration the corresponding quotient classes and quotient set.

Definition 9.11. *$\mathcal{R}_{CL}$ is the category whose objects and maps are defined as follows:*

$$Ob(\mathcal{R}_{CL}) := \{\varphi | \varphi \text{ is a formula of } \mathbf{PA}\}$$

and if $\varphi, \psi \in Ob(\mathcal{R}_{CL})$ then

$$\mathcal{R}_{CL}(\varphi, \psi) := \{e \in \mathbb{N} | e \Vdash \varphi \to \psi\} / \approx .$$

We refer to the equivalent class of a realizer e as $\mathbf{e}$.

[5]We notice that this definition is not decidable! In order to work with a decidable equality one must pass on to type theory.

Similarly, $\mathcal{R}_{IL}$ is the category whose objects and maps are defined as follows:

$$Ob(\mathcal{R}_{IL}) := \{\varphi | \varphi \textit{ is a formula of } \mathbf{HA}\}$$

and if $\varphi, \psi \in Ob(\mathcal{R}_C)$ then

$$\mathcal{R}_{IL}(\varphi, \psi) := \{e \in \mathbb{N} | e \Vdash \varphi \to \psi\} / \approx .$$

We refer to the equivalent class of a realizer e as $\mathbf{e}$.

For convenience, we will write only $\mathcal{R}$ to speak at the same time about both $\mathcal{R}_{IL}$ and $\mathcal{R}_{CL}$, supposing we are working with an appropriate language $\mathcal{L}$, even though we will find that the whole problem we are going to tackle from now up to the end of this chapter makes sense only in the intuitionistic case (and we could expect this, since we know from Theorem (9.4) that the rules of $CL_=$ are not valid in Kleene's realizability). From the definition we gather that the set $\mathcal{R}(\varphi, \psi)$ is a singleton if and only if there is a realizer from φ to ψ. The fact that $\mathcal{R}$ gives indeed rise to a category can be shown really easily. The identity map in $\mathcal{R}$ is given by the class of realizers that are programs that returns as output the input they have received. Given $\mathbf{e} \Vdash \varphi \to \psi$ and $\mathbf{e}' \Vdash \psi \to \chi$, then $Ap(\mathbf{e}', Ap(\mathbf{e}, -)) \Vdash \varphi \to \chi$. Hence we can define composition and the identity map in $\mathcal{R}$. Trivially, identity and associativity are respected. Thus, $\mathcal{R}$ is indeed a category.

$\mathcal{R}$ has $\bot$ and $\top$ as initial and terminal object respectively: given a formula φ, any program that gives a realizer for it is a realizer for $\bot \to \varphi$ since it gives a realizer for φ starting from a realizer for $\bot$, that is nothing; any terminating program that takes a realizer for φ as input is a map $\varphi \to \top$, since everything is a realizer for $\top$. $\mathcal{R}$ has also all binary products: given φ and ψ formulas, then $\pi_\mathbf{1} \Vdash \varphi \wedge \psi \to \varphi$ and $\pi_\mathbf{2} \Vdash \varphi \wedge \psi \to \psi$, and for every map $\mathbf{e} \Vdash \chi \to \varphi$, $\mathbf{d} \Vdash \chi \to \psi$ there is the unique map $\langle \mathbf{e}, \mathbf{d} \rangle \Vdash \chi \to \varphi \wedge \psi$ in $\mathcal{R}$ making the following product diagram commute:

$$\begin{tikzcd}
& \chi \arrow[ddl, "\mathbf{e}"'] \arrow[d, dashed, "\langle \mathbf{e},\mathbf{d} \rangle"] \arrow[ddr, "\mathbf{d}"] & \\
& \varphi \wedge \psi \arrow[dl, "\pi_\mathbf{1}"] \arrow[dr, "\pi_\mathbf{2}"'] & \\
\varphi & & \psi
\end{tikzcd}$$

One may wonder if the adjunctions in Theorem (6.2) and in Theorem (6.3) still hold. If this is the case, then we would be able to describe in detail how realizers change from simpler propositions to more complex ones. In order to reach this goal, we will stick to the following plan: we will analyze how realizers of a set proposition can be built starting from the realizers of the simpler parts it is formed, following the inductive Definition (9.7); then we will define reasonable functors, trying to mimic the situation of Theorem (6.2) and Theorem (6.3).

First of all, we can define in the special case of the category $\mathcal{R}$ of realizers the trivial functor !:

$$\begin{array}{cccc}
! : & \mathcal{R} & \longrightarrow & \mathbf{1} \\
 & \varphi & \mapsto & \bullet \\
 & (\mathbf{e} \Vdash \varphi \to \psi) & \mapsto & (1_\bullet : \bullet \to \bullet)
\end{array}$$

and the diagonal functor Δ:

$$\begin{array}{cccc}
\Delta : & \mathcal{R} & \longrightarrow & \mathcal{R} \times \mathcal{R} \\
 & \varphi & \mapsto & (\varphi, \varphi) \\
 & (\mathbf{e} \Vdash \varphi \to \psi) & \mapsto & ((\mathbf{e}, \mathbf{e}) \Vdash (\varphi, \varphi) \to (\psi, \psi))
\end{array}$$

Quite easily, we can mimic from Definition (6.1) the construction of a true $\top$ and a false $\bot$ functors as follows:

$$\begin{array}{cccc}
\bot & \mathbf{1} & \longrightarrow & \mathcal{R} \\
 & \bullet & \mapsto & \bot \\
 & (\bullet \to \bullet) & \mapsto & (\mathbf{1}_\bot \Vdash \bot \to \bot)
\end{array}
\qquad
\begin{array}{cccc}
\top : & \mathbf{1} & \longrightarrow & \mathcal{R} \\
 & \bullet & \mapsto & \top \\
 & (\bullet \to \bullet) & \mapsto & (\mathbf{1}_\top \Vdash \top \to \top)
\end{array}$$

where $1_\bot$ and $1_\top$ are functional programs that give as outputs the same inputs they receive.

Proposition 9.2. *In the case of the category $\mathcal{R}$ of realizability, the false functor $\perp$ and the true functor $\top$ are respectively left and right adjoint to the trivial functor $!$:*

$$\perp\vdash !\vdash\top$$

Proof. Let $\varphi \in Ob(\mathbf{R})$. There is an isomorphism $\mathbf{1}(!(\varphi),\bullet) \cong \mathcal{R}(\varphi,\top(\bullet))$, that is an isomorphism $\mathbf{1}(\bullet,\bullet) \cong \mathcal{R}(\varphi,\top)$, sending the identity map $1_\bullet$ into the identity map $\mathbf{1}_\varphi$ in $\mathcal{R}$ (and this is obviously a realizer for $\top$). It is trivially natural in $\bullet$ and φ. There is another isomorphism $\mathcal{R}(\perp(\bullet),\varphi) \cong \mathbf{1}(\bullet,!(\varphi))$, that is an isomorphism $\mathcal{R}(\perp,\varphi) \cong \mathbf{1}(\bullet,\bullet)$, sending any constant map into the identity map of $\mathbf{1}$. It is trivially natural in $\bullet$ and φ. □

From Kleene's realizability, we know that a realizer of a conjunction is the pair obtained from the realizers of the two propositions that are part of the conjunction. Let's suppose that φ,ψ, φ' and ψ' are propositions such that $\mathbf{e} \Vdash \varphi \to \psi$ and $\mathbf{d} \Vdash \varphi' \to \psi'$. A realizer for $\varphi\wedge\psi \to \varphi'\wedge\psi'$ is a functional program sending a realizer of $\varphi\wedge\psi$ into a realizer of $\varphi'\wedge\psi'$, which is a couple where the first component is a realizer of φ' and the second is a realizer of ψ'. If $\mathbf{z} \Vdash \varphi\wedge\psi$, then $\pi_1(\mathbf{z}) \Vdash \varphi$ and $\pi_2(\mathbf{z}) \Vdash \psi$. Hence $\mathbf{e}(\pi_1(\mathbf{z})) \Vdash \varphi'$ and $\mathbf{d}(\pi_2(\mathbf{z})) \Vdash \psi'$, and we can form a realizer for $\varphi'\wedge\psi'$ by pairing $\langle \mathbf{e}(\pi_1(\mathbf{z})), \mathbf{d}(\pi_2(\mathbf{z}))\rangle$. It follows that it is possible to define the following conjunction functor $\wedge$:

$$\begin{array}{cccc} (-)\wedge(-): & \mathcal{R}\times\mathcal{R} & \longrightarrow & \mathcal{R} \\ & (\varphi,\psi) & \mapsto & \varphi\wedge\psi \\ & (\mathbf{e}\Vdash\varphi\to\varphi', \mathbf{d}\Vdash\psi\to\psi') & \mapsto & (\lambda\mathbf{z}.\langle\{\mathbf{e}\}(\pi_1(\mathbf{z})),\{\mathbf{d}\}(\pi_2(\mathbf{z}))\rangle \Vdash \varphi\wedge\psi\to\varphi'\wedge\psi') \end{array}$$

In particular, we can define the conjunction functor when the second proposition of the conjunction is fixed in this way:

$$\begin{array}{cccc} (-)\wedge(\mathfrak{pr}): & \mathcal{R} & \longrightarrow & \mathcal{R} \\ & \varphi & \mapsto & \varphi\wedge\mathfrak{pr} \\ & (\mathbf{e}\Vdash\varphi\to\varphi') & \mapsto & (\lambda\mathbf{z}.\langle\{\mathbf{e}\}(\pi_1(\mathbf{z})),\pi_2(\mathbf{z})\rangle \Vdash \varphi\wedge\mathfrak{pr}\to\varphi'\wedge\mathfrak{pr}) \end{array}$$

Proposition 9.3. *In the case of the category $\mathcal{R}$ of realizability, the conjunction functor $\wedge$ is right adjoint to the diagonal functor Δ:*

$$\Delta\vdash(-)\wedge(-)$$

Proof. Let $(\varphi,\psi) \in Ob(\mathcal{R}\times\mathcal{R})$. There is a map $\Delta(\varphi\wedge\psi) = (\varphi\wedge\psi,\varphi\wedge\psi) \to (\varphi,\psi)$ in $\mathcal{R}\times\mathcal{R}$: it suffices to take $(\lambda\mathbf{z}.\pi_1(\mathbf{z}), \lambda\mathbf{z}.\pi_2(\mathbf{z}))$ as realizer, since trivially $\lambda\mathbf{z}\pi_1(\mathbf{z}) \Vdash \varphi\wedge\psi\to\varphi$ and $\lambda\mathbf{z}\pi_2(\mathbf{z}) \Vdash \varphi\wedge\psi\to\psi$.
Let $\alpha \in Ob(\mathcal{R})$ and suppose that there is a map $(\alpha,\alpha) \to (\varphi,\psi)$ in $\mathcal{R}\times\mathcal{R}$, therefore there are realizers $\mathbf{e} \Vdash \alpha\to\varphi$ and $\mathbf{d} \Vdash \alpha\to\psi$. Given that, we can construct a functional program that produces a realizer for $\varphi\wedge\psi$ from a realizer of α: $\lambda\mathbf{z}.\langle\{\mathbf{e}\}(\mathbf{z}),\{\mathbf{d}\}(\mathbf{z})\rangle \Vdash \alpha\to\varphi\wedge\psi$. Owing to the definition of maps in $\mathcal{R}$ through equivalent classes, the map we have just found is unique. □

A realizer of a disjunction is a pair with first component either 0 or 1 and second component a realizer for either the first or the second proposition in the disjunction. Let's suppose that φ,ψ, φ' and ψ' are propositions such that $\mathbf{e} \Vdash \varphi\to\psi$ and $\mathbf{d} \Vdash \varphi'\to\psi'$. A realizer for $\varphi\vee\psi\to\varphi'\vee\psi'$ is a functional program sending a realizer of $\varphi\vee\psi$ into a realizer of $\varphi'\vee\psi'$, which is a couple where the first component is either 0 or 1 and the second component is a realizer of either φ' or ψ'. If $\mathbf{z} \Vdash \varphi\vee\psi$, then there are two possibilities: either $\pi_1(\mathbf{z}) = 0$ and $\pi_2(\mathbf{z}) \Vdash \varphi$, or $\pi_2(\mathbf{z}) = 1$ and $\pi_2(\mathbf{z}) \Vdash \psi$. In the former, $\{\mathbf{e}\}(\pi_2(\mathbf{z}))$ gives a realizer for φ'; in the latter, $\{\mathbf{d}\}(\pi_2(\mathbf{z}))$ gives a realizer for ψ'. Therefore a realizer for $\varphi'\vee\psi'$ is either $\langle 0,\{\mathbf{e}\}(\pi_2(\mathbf{z}))\rangle$ or $\langle 1,\{\mathbf{d}\}(\pi_2(\mathbf{z}))\rangle$, depending on the form of z. More briefly we can write:

$$\langle\pi_1(\mathbf{z}),\{(1-\pi_1(\mathbf{z}))\cdot\mathbf{e}+\pi_1(\mathbf{z})\cdot\mathbf{d}\}(\pi_2(\mathbf{z}))\rangle \Vdash \varphi'\vee\psi'$$

It follows that it is possible to define the following disjunction functor $\vee$:

$$\begin{array}{cccc} (-)\vee(-): & \mathcal{R}\times\mathcal{R} & \longrightarrow & \mathcal{R} \\ & (\varphi,\psi) & \mapsto & \varphi\vee\psi \\ & (\mathbf{e}\Vdash\varphi\to\varphi', \mathbf{d}\Vdash\psi\to\psi') & \mapsto & (\lambda\mathbf{z}.\langle\pi_1(\mathbf{z}),\{(1-\pi_1(\mathbf{z}))\cdot\mathbf{e}+\pi_1(\mathbf{z})\cdot\mathbf{d}\}(\pi_2(\mathbf{z}))\rangle \Vdash \varphi'\vee\psi') \end{array}$$

Proposition 9.4. *In the case of the category $\mathcal{R}$ of realizability, the disjunction functor $\vee$ is left adjoint to the diagonal functor Δ:*

$$(-) \vee (-) \vdash \Delta$$

Proof. Let $(\varphi, \psi) \in Ob(\mathcal{R} \times \mathcal{R})$. There is a map $(\varphi, \psi) \to \Delta(\varphi \vee \psi) = (\varphi \vee \psi, \varphi \vee \psi)$ in $\mathcal{R} \times \mathcal{R}$: it suffices to take $(\lambda \mathbf{z}.\langle 0, \mathbf{z}\rangle, \lambda \mathbf{z}.\langle 1, \mathbf{z}\rangle)$ as realizer, since trivially $\lambda \mathbf{z}.\langle 0, \mathbf{z}\rangle \Vdash \varphi \to \varphi \vee \psi$ and $\lambda \mathbf{z}.\langle 1, \mathbf{z}\rangle \Vdash \psi \to \varphi \vee \psi$.
Let $\alpha \in Ob(\mathcal{R})$ and suppose that there is a map $(\varphi, \psi) \to (\alpha, \alpha)$ in $\mathcal{R} \times \mathcal{R}$, thus there are realizers $\mathbf{e} \Vdash \varphi \to \alpha$ and $\mathbf{d} \Vdash \psi \to \alpha$. Given that, we can construct a functional program that produces a realizer for α from a realizer of $\varphi \vee \psi$: $\lambda \mathbf{z}.\{(1 - \pi_1(\mathbf{z})) \cdot \mathbf{e} + \pi_2(\mathbf{z}) \cdot \mathbf{d}\}\pi_2(\mathbf{z}) \Vdash \varphi \vee \psi \to \alpha$. Owing to the definition of maps in $\mathcal{R}$ through equivalent classes, the map we have just found is unique. □

Let's consider the case of the implication functor $(-)^{\mathfrak{pr}}$ and let's suppose that we have a realizer $\mathbf{e} \Vdash \varphi \to \varphi'$. We would like to find a realizer for $(\mathfrak{pr} \to \varphi) \to (\mathfrak{pr} \to \varphi')$, thus a functional program producing realizers for $(\mathfrak{pr} \to \varphi')$ out of realizers for $(\mathfrak{pr} \to \varphi)$. If $\mathbf{z} \Vdash \mathfrak{pr} \to \varphi$ and $\mathbf{w} \Vdash \mathfrak{pr}$, then $\mathbf{e}(\mathbf{z}(\mathbf{w}))$ is a realizer for φ'. It follows that it is possible to define the implication functor $(-)^{\mathfrak{pr}}$:

$$\begin{array}{cccc} (-)^{\mathfrak{pr}}: & \mathcal{R} & \longrightarrow & \mathcal{R} \\ & \varphi & \mapsto & \mathfrak{pr} \to \varphi \\ & (\mathbf{e} \Vdash \varphi \to \varphi') & \mapsto & (\lambda \mathbf{w}.\lambda \mathbf{z}.\{\mathbf{e} \circ \mathbf{z}\}(\mathbf{w}) \Vdash (\mathfrak{pr} \to \varphi) \to (\mathfrak{pr} \to \varphi')) \end{array}$$

Proposition 9.5. *In the case of the category $\mathcal{R}$ of realizability, the conjunction of $\mathfrak{pr}$ functor $(-) \wedge \mathfrak{pr}$ is left adjoint to the implication of $\mathfrak{pr}$ functor $(-)^{\mathfrak{pr}}$:*

$$(-) \vee (\mathfrak{pr}) \vdash (-)^{\mathfrak{pr}}$$

Proof. Let $\varphi \in Ob(\mathcal{R})$. There is a map $\varphi \to ((\varphi) \wedge \mathfrak{pr})^{\mathfrak{pr}} = (\mathfrak{pr} \to (\varphi \wedge \mathfrak{pr}))$ in $\mathcal{R}$: it suffices to take $\lambda \mathbf{z}.\lambda \mathbf{w}.\langle \mathbf{z}, \mathbf{w}\rangle$ as its realizer.
Let $\alpha \in Ob(\mathcal{R})$ and suppose that there is a map $\mathbf{e} \Vdash \varphi \to (\mathfrak{pr} \to \alpha)$. We need to find a realizer for $(\varphi \wedge \mathfrak{pr}) \to \alpha$. Let $\mathbf{z}$ be a realizer of $\varphi \wedge \mathfrak{pr}$, so $\pi_1(\mathbf{z}) \Vdash \varphi$ and $\pi_2(\mathbf{z}) \Vdash \mathfrak{pr}$. It follows that $\{\mathbf{e}\}(\pi_1(z)) \Vdash \mathfrak{pr} \to \alpha$ and also $\{\mathbf{e}\}(\pi_1(z))(\pi_2(z)) \Vdash \alpha$. In conclusion: $\lambda \mathbf{z}.\{\mathbf{e}\}(\pi_1(\mathbf{z}))(\pi_2(\mathbf{z})) \Vdash (\varphi \wedge \mathfrak{pr}) \to \alpha$. Owing to the definition of maps in $\mathcal{R}$ through equivalent classes, the map we have just found is unique. □

Let's suppose $\mathbf{e}$ to be a realizer for the proposition $\varphi \to \varphi'$. Given that, we would like to find a realizer for the proposition $\neg\varphi' \to \neg\varphi$, i.e. for the proposition $(\varphi' \to \bot) \to (\varphi \to \bot)$. If $\mathbf{z} \Vdash \varphi$ then applying $\{e\}$ to it gives a realizer for φ', hence $\{\mathbf{w} \circ \mathbf{e}\}(\mathbf{z})$ is a realizer for $\bot$ everytime $\mathbf{w} \Vdash \varphi' \to \bot$. It follows that it is possible to define the negation functor $\neg(-)$:

$$\begin{array}{cccc} \neg(-): & \mathcal{R} & \longrightarrow & \mathcal{R}^{op} \\ & \varphi & \mapsto & \neg\varphi \\ & (\mathbf{e} \Vdash \varphi \to \varphi') & \mapsto & (\lambda \mathbf{w}.\lambda \mathbf{z}.\{\mathbf{w} \circ \mathbf{e}\}(\mathbf{z}) \Vdash \neg\varphi' \to \neg\varphi) \end{array}$$

Proposition 9.6. *In the case of the category $\mathcal{R}$ of realizability, the negation functor $\neg(-)$ is left adjoint to the opposite negation functor $\neg^{op}(-)$:*

$$\neg(-) \vdash \neg^{op}(-)$$

Proof. Let $\varphi \in Ob(\mathcal{R})$. There is a map $\varphi \to \neg^{op}(\neg(\varphi)) = \neg\neg\varphi$ in $\mathcal{R}$, that is a map $\varphi \to ((\varphi \to \bot) \to \bot)$: it is easy to see that $\lambda \mathbf{w}.\lambda \mathbf{z}.\{\mathbf{w} \circ \mathbf{e}\}(\mathbf{z})$ realizes it.
Let $\alpha \in Ob(\mathcal{R})$ and suppose that there is a map $\mathbf{e} \Vdash (\varphi \to \neg\alpha) = (\varphi \to (\alpha \to \bot))$. We need to show that there exists a unique map $\neg\varphi \to \alpha$ in $\mathcal{R}^{op}$. So we need to find a realizer for $\alpha \to (\varphi \to \bot)$. If $\mathbf{z} \Vdash \alpha$ and $\mathbf{w} \Vdash \varphi$, then $\{\mathbf{e}\}(\mathbf{w}) \Vdash \alpha \to \bot$ and $\{\mathbf{e}\}(\mathbf{w})(\mathbf{z}) \Vdash \bot$. Concluding: $\lambda \mathbf{z}.\lambda \mathbf{w}.\{\mathbf{e}\}(\mathbf{w})(\mathbf{z}) \Vdash \alpha \to \neg\varphi$. Owing to the definition of maps in $\mathcal{R}$ through equivalent classes, the map we have just found is unique. □

Summing up all we have discovered so far, we can formulate the following result, which is the correspondent of the Theorem (6.2) in the case of the category $\mathcal{R}$ of realizability.

Theorem 9.5. *Let $\mathcal{L}$ be a language for the intuitionistic propositional logic IL and let $\mathfrak{pr}$ be a fixed proposition. Let $\mathcal{R}$ be the category of realizability. Then, in the notation introduced above:*

1. *the true functor $\top$ is right adjoint to the trivial functor $!$:*
$$! \dashv \top$$
2. *the false functor $\bot$ is left adjoint to the trivial functor $!$:*
$$\bot \dashv !$$
3. *the conjunction functor $(-) \wedge (-)$ is right adjoint to the diagonal functor Δ:*
$$\Delta \dashv (-) \wedge (-)$$
4. *the disjunction functor $(-) \vee (-)$ is left adjoint to the diagonal functor Δ:*
$$(-) \vee (-) \dashv \Delta$$
5. *the implication of $\mathfrak{pr}$ functor $(-)^{\mathfrak{pr}}$ is right adjoint to the conjunction of $\mathfrak{pr}$ functor $(-) \wedge \mathfrak{pr}$:*
$$(-) \wedge \mathfrak{pr} \dashv (-)^{\mathfrak{pr}}$$
6. *the negation functor $\neg(-)$ is left adjoint to the opposite negation functor $\neg^{op}(-)$:*
$$\neg(-) \dashv \neg^{op}(-)$$

One can now wonder whether the above theorem could be extended to the case of classical logic or not. In order to formulate a corresponding classical result, we should be able to show that there is an adjunction
$$\neg^{op}(-) \vdash \neg(-)$$
and so that $\neg^{op}(-) : \mathcal{R}^{op} \longrightarrow \mathcal{R}$ is an equivalence. Apparently, this is not possible at all: Theorem (9.4) states clearly that the rules of CL are not valid in Kleene's realizability, and this is true even for the propositional logic, since the rule $\to_R$ of the classical implication on the right has no realizer in **PA**.

More specifically, if $\neg^{op}(-) \vdash \neg(-)$ then there should be an equivalence
$$\mathcal{R}(\neg\varphi, \psi) \cong \mathcal{R}^{op}(\varphi, \neg\psi)$$
for every formula φ and ψ. In particular, for $\varphi \equiv \neg\psi$, we should find a Kleene's realizer for the formula $\neg\neg\psi \to \psi$ if and only if $\neg\psi \to \neg\psi$ has one, which is actually trivial (it suffices to take the identity function $\lambda x.x$). Now, if $\neg\neg\psi \to \psi$ is realized for every formula ψ, then there must a realizer
$$\mathbf{e} \Vdash \neg\neg(A \vee \neg A) \to A \vee \neg A$$
For sure, we can produce a realizer for the proposition $\neg\neg(A \vee \neg A)$, no matter what A is, because we can simply take the function with no output $\Lambda\vec{x}.\bot$. For $A \equiv \exists y.\chi_{T(e,x,y)}(e,x,y) = 1$, let's denote this realizer $\mathbf{k} \Vdash \neg\neg(\exists y.\chi_{T(e,x,y)}(e,x,y) = 1 \vee (\exists y.\chi_{T(e,x,y)}(e,x,y) = 1 \to \bot))$. Thus we find a realizer
$$\{\mathbf{e}\}(\mathbf{k}) \Vdash \exists y.\chi_{T(e,x,y)}(e,x,y) = 1 \vee (\exists y.\chi_{T(e,x,y)}(e,x,y) = 1 \to \bot)$$
and so we would be able to solve the Halting problem, which is not possible.

Theorem 9.6. *Let $\mathcal{L}$ be a language for the classical propositional logic CL. Let $\mathcal{R}$ be the category of realizability. In the notation introduced above, there is no adjuntion*
$$\neg^{op}(-) \nvdash \neg(-)$$

Proof. If this is the case, then we would be able to solve the Halting problem, as we showed above[6]. □

Let's extend now the language $\mathcal{L}$ to a predicative one and explore the predicative case. As we did before, we can define the category of realizability $\mathcal{R}$, including this time all possible formulas with quantified variables too. We introduce the notation $\mathcal{R}(x_1, ...x_n)$ in order to indicate the subcategory of $\mathcal{R}$ where formulas contain free variables among $x_1, ..., x_n$.

Let $\vec{x} = x_1, ..., x_n$. From Kleene's realizability, we know that a realizer of the formula $\exists y.\varphi(\vec{x}, y)$ is a natural number n such that $\pi_2(n) \Vdash \varphi[\pi_1(n)/y]$. Let's suppose that $\mathbf{e}$ is a realizer and $\varphi(\vec{x}, y)$, $\psi(\vec{x}, y)$ are formulas with free variables among $x_1, ..., x_n, y$, such that:

$$\mathbf{e} \Vdash \varphi(\vec{x}, y) \to \psi(\vec{x}, y)$$

A realizer for $\exists y.\varphi(\vec{x}, y) \to \exists y.\psi(\vec{x}, y)$ is a functional program sending a realizer of $\exists y.\varphi(\vec{x}, y)$, say $\mathbf{k} \Vdash \exists y.\varphi(\vec{x}, y)$, into a realizer of $\exists y.\psi(\vec{x}, y)$. By definition then: $\pi_2(\mathbf{k}) \Vdash \varphi[\pi_1(\mathbf{k})/y]$. Hence: $\{\mathbf{e}\}(\pi_2(\mathbf{k})) \Vdash \psi[\pi_1(\mathbf{k})/y]$, and so $\lambda\mathbf{k}.\langle\pi_1(\mathbf{k}), \{\mathbf{e}\}(\pi_2(\mathbf{k}))\rangle \Vdash \exists y.\varphi(\vec{x}, y) \to \exists y.\psi(\vec{x}, y)$. We can define the existential quantification functor:

$$\begin{array}{llcl} \exists_y : & \mathcal{R}(x_1, ..., x_n, y) & \longrightarrow & \mathcal{R}(x_1, ..., x_n) \\ & \varphi(\vec{x}, y) & \mapsto & \exists y.\varphi(\vec{x}, y) \\ & (\mathbf{e} \Vdash \varphi(\vec{x}, y) \to \psi(\vec{x}, y)) & \mapsto & (\lambda\mathbf{k}.\langle\pi_1(\mathbf{k}), \{\mathbf{e}\}(\pi_2(\mathbf{k}))\rangle \Vdash \exists y.\varphi(\vec{x}, y) \to \exists y.\psi(\vec{x}, y)) \end{array}$$

Also, we can define the weakening functor of the context of the variables $\vec{x}$ with extra variable y as follows:

$$\begin{array}{llcl} H[\vec{x}/\vec{x}] : & \mathcal{R}(x_1, ..., x_n) & \longrightarrow & \mathcal{R}(x_1, ..., x_n, y) \\ & \varphi(\vec{x}) & \mapsto & \varphi(\vec{x}, y) \\ & (\mathbf{e} \Vdash \varphi(\vec{x}) \to \psi(\vec{x})) & \mapsto & (\mathbf{e} \Vdash \varphi(\vec{x}, y) \to \psi(\vec{x}, y)) \end{array}$$

Proposition 9.7. *In the case of the category $\mathcal{R}$ of realizability, the existential quantification functor $\exists_y$ is left adjoint to the weakening functor $H[\vec{x}/\vec{x}]$ of the context of the variables $\vec{x}$ with extra variable y:*

$$\exists_y \vdash H[\vec{x}/\vec{x}]$$

Proof. Let $\varphi(\vec{x}, y) \in Ob(\mathcal{R}(\vec{x}, y))$. There is a map $\varphi(\vec{x}, y) \leq \exists y.\varphi(\vec{x}, y)$ in $\mathcal{R}(\vec{x}, y)$ given by the equivalent class of the realizers $\lambda\mathbf{k}.\langle y, \mathbf{k}\rangle$. Let now $\alpha(\vec{x}) \in Ob(\mathcal{R}(\vec{x}))$ and let $\mathbf{e} \Vdash \varphi(\vec{x}, y) \to \alpha(\vec{x})$ be a map in $\mathcal{R}(\vec{x}, y)$. We need to show that there exists a unique map $\exists y.\varphi(\vec{x}, y) \to \alpha(\vec{x})$, so we have to find a realizer for $\alpha(\vec{x})$, given a realizer $\mathbf{w}$ of $\exists y.\varphi(\vec{x}, y)$. By definition: $\pi_2(\mathbf{w}) \Vdash \varphi[\pi_1(\mathbf{w})/y]$, therefore $\{\mathbf{e}\}(\pi_2(\mathbf{w})) \Vdash \alpha(\vec{x})$. Concluding, we have found a map $\lambda\mathbf{w}.\{\mathbf{e}\}(\pi_2(\mathbf{w})) \Vdash \exists y.\varphi(\vec{x}, y) \to \alpha(\vec{x})$, which is unique due to the definition of maps in $\mathcal{R}$ by means of equivalent classes. □

From Kleene's realizability, we know that a realizer of the formula $\forall y.\varphi(\vec{x}, y)$ is a natural number n such that $\forall k.(\{n\}(k) \Vdash \varphi[k/y])$. Let's suppose that $\mathbf{e}$ is a realizer and $\varphi(\vec{x}, y)$, $\psi(\vec{x}, y)$ are formulas with free variables among $x_1, ..., x_n, y$, such that:

$$\mathbf{e} \Vdash \varphi(\vec{x}, y) \to \psi(\vec{x}, y)$$

A realizer for $\forall y.\varphi(\vec{x}, y) \to \forall y.\psi(\vec{x}, y)$ is a functional program sending a realizer of $\forall y.\varphi(\vec{x}, y)$, say $\mathbf{k} \Vdash \forall y.\varphi(\vec{x}, y)$, into a realizer of $\forall y.\psi(\vec{x}, y)$. By definition then: $\forall k.(\{\mathbf{w}\}(k) \Vdash \varphi[k/y])$.

[6] To be precise, the proof of the Theorem holds as long as the language $\mathcal{L}$ is previously extended to a predicative language, by introduction of quantification on a countable set of variables. However, the possibility of solving the Halting problem shows that there is no adjunction $\neg^{op}(-) \nvdash \neg(-)$ not only in the case of a predicative language, but also in the case of a propositional language.

Hence: $\{\mathbf{e}\}(\{\mathbf{w}\}(k)) \Vdash \psi[k/y]$, and so $\forall k.(\{\mathbf{e}\}(\{\mathbf{w}\}(k)) \Vdash \psi[k/y])$ and $\lambda\mathbf{w}.\mathbf{e}\circ\mathbf{w} \Vdash \forall y.\varphi(\vec{x},y) \to \forall y.\psi(\vec{x},y)$. We can define the universal quantification functor:

$$\begin{array}{cccc} \forall_y : & \mathcal{R}(x_1,...,x_n,y) & \longrightarrow & \mathcal{R}(x_1,...,x_n) \\ & \varphi(\vec{x},y) & \mapsto & \forall y.\varphi(\vec{x},y) \\ & (\mathbf{e} \Vdash \varphi(\vec{x},y) \to \psi(\vec{x},y)) & \mapsto & (\lambda\mathbf{w}.\mathbf{e}\circ\mathbf{w} \Vdash \forall y.\varphi(\vec{x},y) \to \forall y.\psi(\vec{x},y)) \end{array}$$

Proposition 9.8. *In the case of the category $\mathcal{R}$ of realizability, the universal quantification functor $\forall_y$ is right adjoint to the weakening functor $H[\vec{x}/\vec{x}]$ of the context of the variables $\vec{x}$ with extra variable y:*

$$H[\vec{x}/\vec{x}] \vdash \forall_y$$

Proof. Let $\varphi(\vec{x},y) \in Ob(\mathcal{R}(\vec{x},y))$. There is a map $\forall y.\varphi(\vec{x},y) \to \varphi(\vec{x},y)$ in $\mathcal{R}(\vec{x},y)$, given by the equivalent class of realizers $\lambda\mathbf{w}.(\lambda y.\{\mathbf{w}\}(y))$. Indeed, if $\mathbf{w} \Vdash \forall y.\varphi(\vec{x},y)$ then $\forall k.(\{\mathbf{w}\}(k) \Vdash \varphi[k/y])$, so $\lambda y.\{\mathbf{w}\}(y) \Vdash \varphi(\vec{x},y)$. Let $\alpha(\vec{x}) \in Ob(\mathcal{R}(\vec{x}))$ and let $\mathbf{e} \Vdash \alpha(\vec{x},y) \to \varphi(\vec{x},y)$. We need to show that there exists a unique map $\alpha(\vec{x}) \to \forall y.\varphi(\vec{x},y)$, so we have to find a realizer for $\forall y.\varphi(\vec{x},y)$, given a realizer $\mathbf{z}$ of $\alpha(\vec{x})$. However, $\mathbf{z}$ is a program depending on free variables among $\vec{x}$, and we can modify it so it depends on free variables among $\vec{x},y$, by adding the blind variable y in the inputs: $\mathbf{z} \Vdash \alpha(\vec{x},y)$. Thus $\{\mathbf{e}\}(\mathbf{z}) \Vdash \varphi(\vec{x},y)$ and $\forall k.(\{\mathbf{e}\}(\mathbf{z}(k)) \Vdash \varphi[k/y])$, which means that $\lambda k.\{\mathbf{e}\}(\mathbf{z}(k)) \Vdash \forall y.\varphi(\vec{x},y)$. Concluding, we have found a map $\lambda\mathbf{z}.(\lambda k.\{\mathbf{e}\}(\mathbf{z}(k))) \Vdash \alpha(\vec{x}) \to \forall y.\varphi(\vec{x},y)$, which is unique due to the definition of maps in $\mathcal{R}$ by means of equivalent classes. □

In the end, we can extend Theorem (9.6) to the predicative case, proving it with the same argument as before.

Theorem 9.7. *Let $\mathcal{L}$ be a language for the classical predicative logic CL. Let $\mathcal{R}$ be the category of realizability, enriched with the existential and universal quantification on variables. In the notation introduced above, there is no adjuntion*

$$\neg^{op}(-) \nvdash \neg(-)$$

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Scoasse, you make me feel safe and at home in such a noisy and dangerous world and you never stop proving me you will always be by my side no matter what happens.

Dani, Pre and Pera, you are my real Power Rangers: somebody I can rely on without any hesitation, somebody I can breath fresh air with.

Forn, you are my eternity-of-being: we met exactly when destiny had scheduled it, but we have been strong enough not to drift apart over time, when we began to grow and build our own lives.

Arianna and Lia, you two have been my life line at university and you are becoming an anchor in my everyday life.

Willy, you and me together have understood that life is more than just meets the eye and have explored some horizons nobody could even dream about.

I cannot tell how many other names I should have cited, but please believe me: it is just a matter of ink and space in the sheet. I affectionately hug you all and thank you for choosing to spend together at least a part of the terrific and surprising adventure that is life.

Only you make my heart beat so strongly and passionately.

Thank you for what you truly, deeply, wonderfully, are.